

# The Octagonal PET II: The Topology of the Limit Sets

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October 2, 2012

## Abstract

This paper is a sequel to [S0]. In [S0], we studied a 1-parameter family of polygon exchange transformations. We showed that this family is completely renormalizable. In this paper, we use the renormalization scheme in [S0] to give a complete classification of topological types of the limit sets which arise in these PETs.

## 1 Introduction

### 1.1 Background

A *polytope exchange transformation* (or PET) is defined by a polytope  $X$  which has been partitioned in two ways into smaller polytopes:

$$X = \bigcup_{i=1}^m A_i = \bigcup_{i=1}^m B_i.$$

For each  $i$  there is some vector  $V_i$  such that  $B_i = A_i + V_i$ . That is, some translation carries  $A_i$  to  $B_i$ . One then defines a map  $f : X \rightarrow X$  by the formula  $f(x) = x + V_i$  for all  $x \in \text{int}(A_i)$ . It is understood that  $f$  is not defined for points in the boundaries of the small polytopes. The inverse map is defined by  $f^{-1}(y) = y - V_i$  for all  $y \in \text{int}(B_i)$ .

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\* Supported by N.S.F. Research Grant DMS-0072607

The simplest examples of PETs are 1-dimensional systems, known as *interval exchange transformations* (or IETs). These systems have been extensively studied in the past 30 years, and there are close connections between IETs and other areas of mathematics such as Teichmüller theory. See, for instance, [Y] and [Z] and the many references mentioned therein.

Some examples of polygon exchange maps have been studied in [AG], [AKT], [H], [Hoo], [LKV], [Low], [S2], [S3], and [T]. Some definitive theoretical work concerning the (zero) entropy of such systems is done in [GH1], [GH2], and [B].

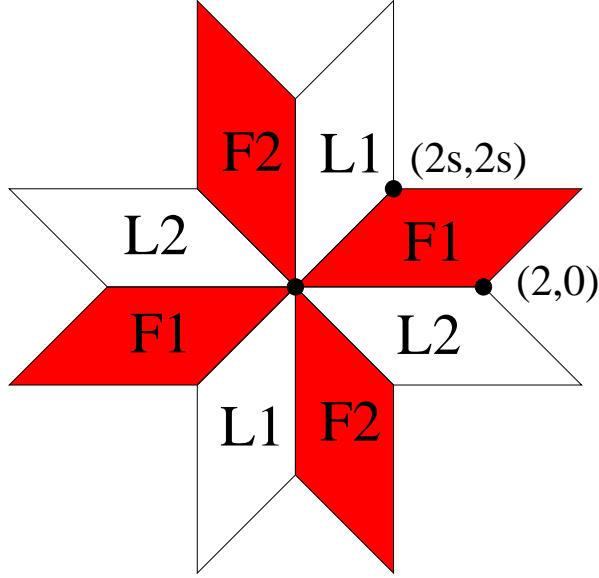
The *Rauzy renormalization* [R] gives a satisfying renormalization theory for the family of IETs all having the same number of intervals in the partition. Often, renormalization phenomena are found in individual PETs, and in the papers [S0], [S1] and [Hoo], there were constructed families of PETs which had a renormalization theory at least vaguely similar to the Rauzy renormalization.

In [S0], we introduced a family of PETs in every even dimension. In dimension  $2n$ , the objects are indexed by  $GL_n(\mathbf{R})$ . In the two-dimensional setting, there is a 1-parameter family. We studied this family in detail, and in particular worked out a renormalization scheme for the family. In this paper, we use the renormalization scheme to work out the topology of the associated limit sets of the 2-dimensional examples.

In [S0] we give a somewhat fuller discussion of the background and context for our examples.

## 1.2 Construction of the PET

Here we recall the basic construction given in [S0], at least in the 2-dimensional case. Our construction depends on a parameter  $s \in (0, \infty)$ . The 8 parallelograms in Figure 1.1 are the orbit of a single parallelogram  $P$  under a dihedral group of order 8. Two of the sides of  $P$  are determined by the vectors  $(2, 0)$  and  $(2s, 2s)$ . For  $j = 1, 2$ , let  $F_j$  denote the parallelogram centered at the origin and translation equivalent to the ones in the picture labeled  $F_j$ . Let  $L_j$  denote the lattice generated by the sides of the parallelograms labeled  $L_j$ . (Either one generates the same lattice.)



**Figure 1.1:** The scheme for the PET.

It turns out that  $F_i$  is a fundamental domain for  $L_j$ , for all  $i, j \in \{1, 2\}$ . Given  $p \in F_j$  we let

$$f'(p) = p + V_p \in F_{3-j}, \quad V_p \in L_{3-j}. \quad (1)$$

The choice of  $V_p$  is almost always unique, on account of  $F_{3-j}$  being a fundamental domain for  $L_{3-j}$ . When the choice is not unique, we leave  $f'$  undefined. When  $p \in F_1 \cap F_2$  we have  $V_p = 0 \in L_1 \cap L_2$ . We define  $f = (f')^2$ , which preserves both  $F_1$  and  $F_2$ , and we set  $X = F_1$ . Our system is  $f : X \rightarrow X$ , which we denote by  $(X, f)$ .

### 1.3 Prior Results

The results in [S0] (and in this paper) ultimately follow from a renormalization scheme associated to our family. Define the *renormalization map*  $R : (0, 1) \rightarrow [0, 1)$  as follows.

- $R(x) = 1/(2x) - \text{floor}(1/(2x))$  if  $x < 1/2$ .
- $R(x) = 1 - x$  if  $x > 1/2$ .

This map will appear in many of our results. We explain its direct connection to the PETs in §2.

A *periodic tile* for  $(X, f)$  as a maximal convex polygon on which  $f$  and its iterates are completely defined and periodic. We call the union  $\Delta$  of the periodic tiles the *tiling*. In [S0] we proved, among other things, the following results about the tiling.

**Theorem 1.1** *When  $s$  is irrational,  $\Delta_s$  is a full measure set consisting entirely of squares and semi-regular octagons.*

- $\Delta_s$  consists entirely of squares iff  $R^n(s) < 1/2$  for all  $n$ .
- $\Delta_s$  has finitely many octagons iff  $R^n(s) > 1/2$  for finitely many  $n$ .
- $\Delta_s$  has infinitely many octagons iff  $R^n(s) > 1/2$  for infinitely many  $n$ .

The fact that the union of tiles in  $\Delta_s$  has full measure allows us to define the limit set <sup>1</sup>  $\widehat{\Lambda}_s$  to be the set of points  $p$  such that every neighborhood of  $p$  intersects infinitely many tiles of  $\Delta$ .

In [S0] we proved, among other things, the following results about the limit set.

**Theorem 1.2** *Let  $s \in \mathbf{R}$  be irrational. The projection of  $\widehat{\Lambda}_s$  onto any line parallel to an 8th root of unity contains a line segment. Hence  $\widehat{\Lambda}_s$  has Hausdorff dimension at least 1. Moreover,  $\widehat{\Lambda}_s$  is not contained in a finite union of lines.*

We also proved that our 1-parameter family had what one might call hidden hyperbolic symmetry.

**Theorem 1.3** *The Hausdorff dimension of  $\widehat{\Lambda}_s$ , as a function of the parameter, is invariant under the action of the  $(2, 4, \infty)$  hyperbolic reflection triangle group acting on the parameter space by linear fractional transformations.*

The results in [S0] are somewhat more detailed than the ones listed above, and the reader should refer to [S0] for the more detailed versions of the results above.

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<sup>1</sup>For a general polygon exchange map, the limit set is probably best defined as the set of weakly aperiodic points. We call a point  $p \in X$  *weakly aperiodic* if there is a sequence  $\{q_n\}$  converging to  $p$  with the following property. The first iterates of  $f$  are defined on  $q_n$  and the points  $f^k(q_n)$  for  $k = 1, \dots, n$  are distinct.

## 1.4 Topology of the Limit Sets

Here is the main result of this paper.

**Theorem 1.4 (Main)** *Let  $s \in (0, 1)$  be irrational.*

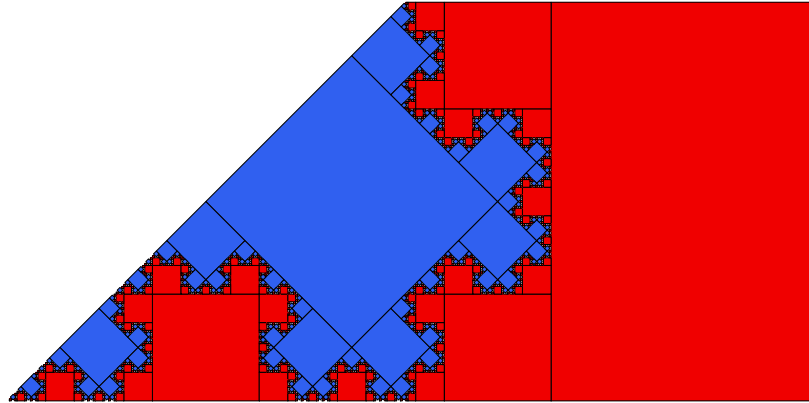
1.  $\widehat{\Lambda}_s$  is a disjoint union of two arcs if and only if  $R^n(s) < 1/2$  for all  $n$ .
2.  $\widehat{\Lambda}_s$  is a finite forest<sup>2</sup> if and only if  $R^n(s) > 1/2$  for finitely many  $n$ .
3.  $\widehat{\Lambda}_s$  is a Cantor set if and only if  $R^n(s) > 1/2$  infinitely often.

Combining the Main Theorem with Theorem 1.1, we get the following corollary.

**Corollary 1.5** *Let  $s \in (0, 1)$  be irrational.*

1.  $\widehat{\Lambda}_s$  is a disjoint union of two arcs if and only if  $\Delta_s$  contains only squares.
2.  $\widehat{\Lambda}_s$  is a finite forest if and only if  $\Delta_s$  contains finitely many octagons.
3.  $\widehat{\Lambda}_s$  is a Cantor set if and only if  $\Delta_s$  contains infinitely many octagons.

Figure 1.2 shows Statement 1 of the Main Theorem in action.

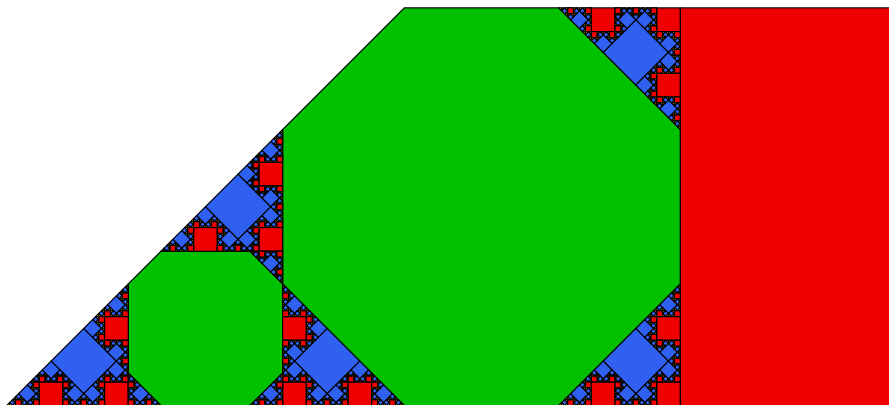


**Figure 1.2:** The left half of  $\Delta_s$  for  $s = \sqrt{3}/2 - 1/2$ .

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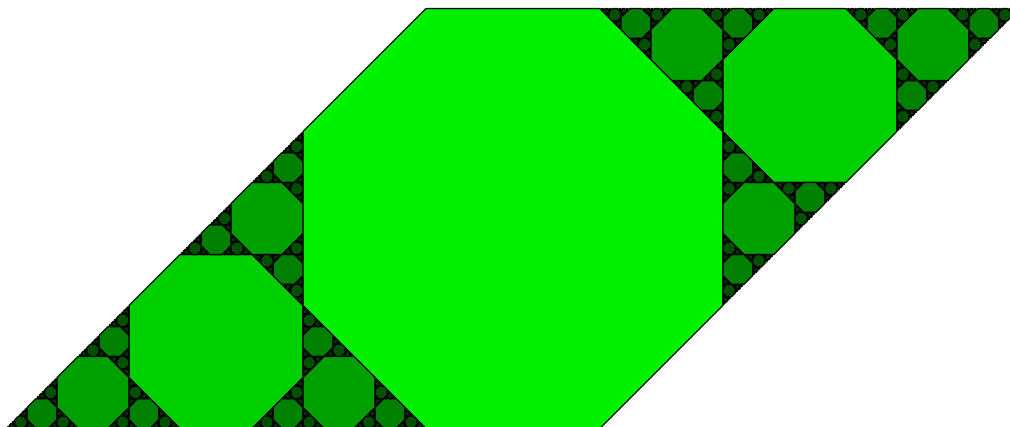
<sup>2</sup>A *finite forest* is a finite disjoint union of finite trees

Figure 1.3 shows Statement 2 of the Main Theorem in action. The parameter is such that its even expansion is  $3, 1, 3, 1, 2, 2, \dots$ . A  $k$  in the  $n$ th position indicates that  $R^n(s) \in (1/(k+1), 1/k)$ .



**Figure 1.3:** The left half of  $\Delta_s$  for  $s = (3, 1, 3, 1, 2, 2, 2\dots)$ .

Figure 1.4 shows Statement 3 of the Main Theorem in



**Figure 1.3:** The left half of  $\Delta_s$  for  $s = \sqrt{2}/2$ .

In all these figures, the right half is a rotated image of the left half. Intuitively, the squares tend to line up in simply connected chunks whose complementary regions are curves and the octagons both split and splice the curves.

This paper is organized as follows. In §2 we will recall the renormalization scheme from [S0], as well as some symmetry results. Following the summary given in §2, the rest of this paper is self-contained. In §3-4 we prove some geometric and combinatorial results about how the tiles of  $\Delta_s$  fill  $X_s$  and how  $\Lambda_s$  intersects various lines of symmetry. In §5-7 we prove the Main Theorem, one statement per chapter.

We would like to say a word about the nature of our proofs. For the sake of giving a readable exposition, there are a number of routine geometric calculations, concerning polygonal regions in the plane, which we omit. Such calculations are all exercises in plane geometry. Rather than write out these calculations, we illustrate them with extensive pictures from our java program OctaPET. It is certainly best to read this paper while using OctaPET; this is how we wrote the paper.

A common mistake made by beginning students is to try to prove a general statement by just considering one example. We do not mean to make this mistake here, even though superficially some of our proofs look like this. In a written paper dealing with a 1-parameter family of systems, we cannot illustrate the picture for every parameter. The pictures we do show are typical for the given parameter interval, and the written arguments we give only make statements which hold for all the relevant parameters.

I would like to thank Nicolas Bedaride, Pat Hooper, Injee Jeong, John Smillie, and Sergei Tabachnikov for interesting conversations about topics related to this work. Some of this work was carried out at ICERM in Summer 2012, and some was carried out during my sabbatical in 2012-13. This sabbatical was funded from many sources. I would like to thank the National Science Foundation, All Souls College, Oxford, the Oxford Maths Institute, the Simons Foundation, the Leverhulme Trust, and Brown University for their support during this time period.

## 2 Preliminaries

### 2.1 Closedness

We start with an essentially obvious result, which we state for the record.

**Lemma 2.1**  $\widehat{\Lambda}_s$  is closed.

**Proof:** Let  $\{p_n\}$  be a sequence of points in  $\widehat{\Lambda}_s$  converging to some  $q$ . We want to show that  $q \in \widehat{\Lambda}_s$ . We need to show that every open neighborhood  $U$  of  $q$  contains infinitely many tiles of  $\Delta_s$ . For some  $n$  we have  $p_n \in U$ . But then some neighborhood  $V$  of  $p_n$  lies in  $U$ . But  $V$  contains infinitely many tiles of  $\Delta_s$ . Hence, so does  $U$ . ♠

### 2.2 The Even Expansion

We call an irrational number  $s \in (0, 1)$  is *oddly even* if it has continued fraction expansion  $[0, a_1, a_2, a_3, \dots]$  with  $a_k$  even for all odd  $k$ . In [S0] we showed that  $s$  is oddly even if and only if  $R^n(s) < 1/2$  for all  $n$ . We call the rational  $s$  *oddly even* if the (finite) orbit  $\{R^n(s)\}$  avoids  $(1/2, 1)$ . In [S0], we proved the following easy-to-believe lemma.

**Lemma 2.2** *Let  $s$  be an oddly even irrational. There is a sequence  $\{r_n\}$  of oddly even rational numbers which converges to  $s$ .*

When  $s$  is irrational, we define the *even expansion* of  $s$  to be  $n_0, n_1, n_2, \dots$  where

$$\frac{1}{n_k + 1} < R^k(s) < \frac{1}{n_k}. \quad (2)$$

Since  $R$  maps each interval  $(1/2n, 1/(2n-1)]$  onto  $[1/2, 1)$ , an irrational  $s \in (0, 1)$  is oddly even if and only if its even expansion consists entirely of even numbers.

We can define the even expansion of  $s \in (0, 1)$  when  $s$  is rational. The last nonzero term in the  $R$ -orbit of  $s$  is  $1/2q$  for some  $q = 1, 2, 3, \dots$  and by convention we take the corresponding term in the even expansion to be  $1/2q$ . The preceding terms lie in the interiors of the intervals in Equation 2. When  $r_n \rightarrow s$  and  $s$  is irrational, even expansions of  $r_n$  converges to the even expansion of  $s$ .



## 2.3 Rational Limits

When we speak of the convergence of compact sets, we mean to use the Hausdorff topology. A sequence of compact sets  $\{P_n\}$  converges to a compact set  $P_\infty$  if, for every  $\epsilon > 0$  there is some  $N$  such that  $n > N$  implies that the Hausdorff distance from  $P_n$  to  $P_\infty$  is less than  $\epsilon$ . The Hausdorff distance between compact sets  $S_1$  and  $S_2$  is the infimal  $d$  such that  $S_j$  is contained in the  $d$ -neighborhood of  $S_{3-j}$  for  $j = 1, 2$ .

Let  $\{s_n\}$  be a sequence of rationals converging to an irrational parameter  $s \in (0, 1)$ . Let  $\Delta_n$  be the periodic tiling associated to  $s_n$  and let  $\Delta_\infty$  be the periodic tiling associated to  $s$ .

**Lemma 2.3 (Approximation)** *Let  $P_\infty$  be a tile of  $\Delta_\infty$ . Then for all large  $n$ , there is a tile  $P_n$  of  $\Delta_n$  such that  $\{P_n\}$  converges to  $P_\infty$ .*

**Proof:** We proved this in [S0, §3,3]. Here is a sketch. When  $s$  is irrational, we show that the periodic points are stable under perturbation. Hence, there is some sequence  $\{P_n\}$  such that  $\lim P_n$  contains an open subset of  $P_\infty$ . We also show that the set of periodic points in  $\Delta_t$  with the same combinatorics (essentially the list of vectors we use when defining the map  $f_n$  and its iterates, written in a parameter-independent basis) is a single tile of  $\Delta_t$ , as long as  $t \neq 1/n$  for  $n = 1, 2, 3, \dots$ . That is, different tiles have different associated dynamical combinatorics. This uniqueness forces  $\lim P_n = P_\infty$ . ♠

**Lemma 2.4** *Let  $P_n$  be a tile of  $\Delta_n$  and  $\{P_n\}$  converges to a polygon  $P_\infty$ . Then  $P_\infty$  is a tile of  $\Delta_\infty$ .*

**Proof:** We give the argument in [S0, §6.2]. Here is a sketch. There is a uniform lower bound on the size of  $P_n$  and a uniform upper bound on the area of  $X_n$ . This puts a uniform upper bound on the period of  $P_n$ . Passing to a subsequence, we can assume that the combinatorics of the orbit is independent of  $n$ . By continuity, any point in  $P_\infty$  is a periodic point for  $f_n$  when  $n$  is sufficiently large. This shows that there is a periodic tile  $P'_\infty$  of  $\Delta_\infty$  which contains  $P_\infty$ . Applying the Convergence Lemma, there is some sequence  $\{P'_n\}$  such that  $P'_n \rightarrow P'_\infty$ . Eventually  $P'_n$  and  $P_n$  overlap, and so we must have  $P_n = P'_n$ . But then  $P'_\infty = P_\infty$ . ♠

We abbreviate the results in this section by saying that the tilings  $\{\Delta_n\}$  converge to the tiling  $\Delta_\infty$ .

## 2.4 Renormalization

In this section we explain the significance of the renormalization map  $R$  which appears in our results.

We use the notation from §1. So,  $F_1 = X$  is the domain of our system and  $F_2$  is the image of  $F_1$  rotated by  $\pi/2$ . The map  $f$  is trivial on the intersection  $F_1 \cap F_2$ . We call this intersection the *trivial tile*.

Suppressing the parameter  $s$ , we define

$$Y = F_1 - F_2 = X - F_2 \subset X. \quad (3)$$

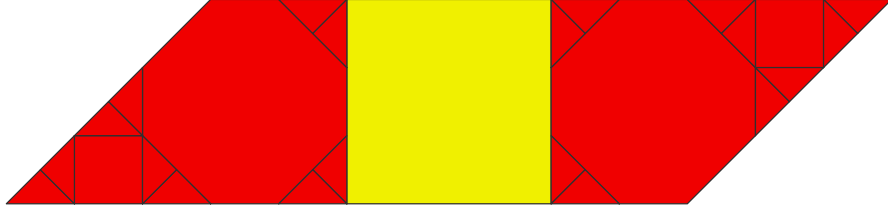
$Y$  is the portion of  $X$  outside the trivial tile.

We call a subset  $S \subset X$  *clean* if  $\partial S$  does not intersect the interior of any tile of  $\Delta$ . Here is the main result from [S0].

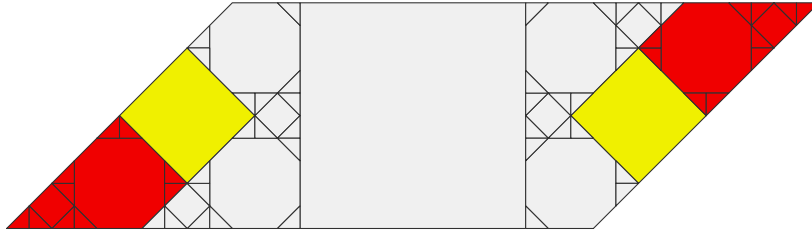
**Theorem 2.5** *Suppose  $s \in (0, 1)$  and  $t = R(s) \in (0, 1)$ . There is a clean set  $Z_s \subset X_s$  and a similarity  $\phi_s : Y_t \rightarrow Z_s$  such that  $\phi_s(\Delta_t \cap Y_t) = \Delta_s \cap Z_s$ .*

1.  $\phi_s$  commutes with reflection in the origin and maps the acute vertices of  $X_t$  to the acute vertices of  $X_s$ .
2. When  $s < 1/2$ , the restriction of  $\phi_s$  to each component of  $Y_t$  is an orientation reversing similarity, with scale factor  $s\sqrt{2}$ .
3. When  $s < 1/2$ , either half of  $\phi_s$  extends to the trivial tile of  $\Delta_t$  and maps it to a tile in  $\Delta_s$ .
4. When  $s < 1/2$ , the only nontrivial orbits which miss  $Z_s$  are contained in the  $\phi_s$ -images of the trivial tile of  $\Delta_t$ . These orbits have period 2.
5. When  $s > 1/2$  the restriction of  $\phi_s$  to each component of  $Y_s$  is a translation.
6. When  $s > 1/2$ , all nontrivial orbits intersect  $Z_s$ .

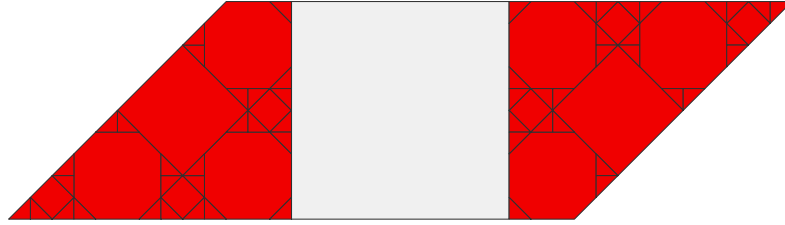
Figures 2.1 and 2.2 illustrate the result for  $s < 1/2$ . Figures 2.3 and 2.4 show the Main Theorem in action for  $s > 1/2$ .



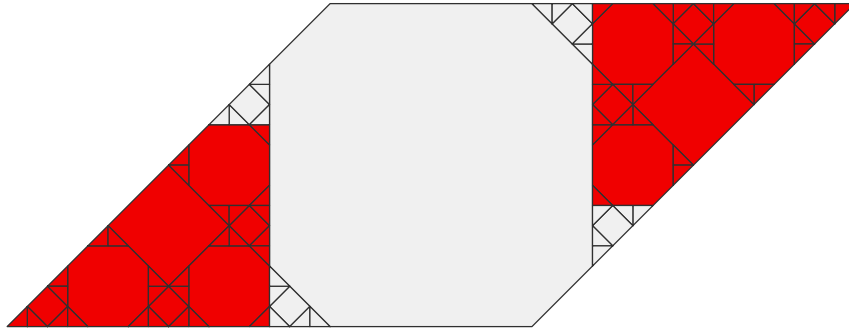
**Figure 2.1:**  $Y_t$  in red for  $t = 3/10 = R(5/13)$ .



**Figure 2.2:**  $Z_s$  in red  $s = 5/13$ .



**Figure 2.3:**  $Y_t$  in red for  $t = R(8/13) = 5/13$ .



**Figure 2.4:**  $Z_s$  in red for  $s = 8/13$ .

## 2.5 Elementary Symmetry

In [S0] we established a number of symmetries of the system. The ones listed in this section have easy proofs, though we do not give them here.

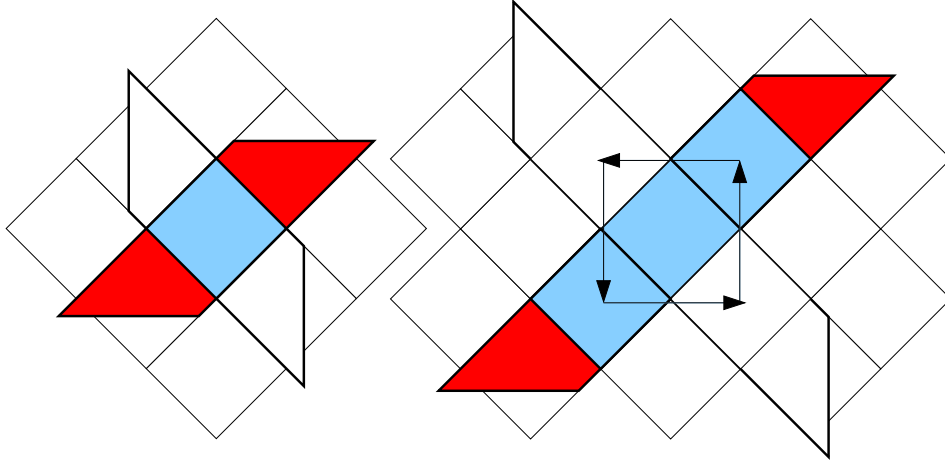
**Rotation Symmetry:**  $\Delta_s$  is invariant under reflection in the origin:

$$\iota(x, y) = (-x, -y), \quad (4)$$

In view of this symmetry, we will often draw only the left half of the picture.

**Inversion Symmetry:** Let  $t = 1/(2s)$ . There is a similarity carrying  $\Delta_s$  to  $\Delta_t$ . The similarity is orientation reversing, fixes the origin, and interchanges lines of slope 0 with lines of slope 1. We called this fact the Inversion Lemma in [S0]. The Inversion Lemma gives shape to the renormalization map  $R$ .

**Central Tiles:** When  $s \in (1/2, 1)$ , the intersection  $F_1 \cap F_2$  is an octagon, which we call the *central tile*. When  $s \leq 1/2$  or  $s \geq 1$ , the intersection  $F_1 \cap F_2$  is a square. This square generates a grid in the plane, and finitely many squares in this grid lie in  $X = F_1$ . We call these squares the *central tiles*. See Figure 2.5.



**Figure 2.5:** The central tiles (blue) for  $s = 5/4$  and  $t = 9/4$ .

For any relevant object  $S \subset X_s$ , the set  $S^0$  denotes the portion of  $S$  lying to the left of the central tiles of  $\Delta_s$ . We will use this notational convention repeatedly.

## 2.6 Insertion Symmetry

When  $s \geq 1$  and  $t = s + 1$ , the intersections  $\Delta_s^0 = \Delta_s \cap X_s^0$  and  $\Delta_t^0 \Delta_t \cap X_s^0$  are isometric. (We give the argument in [S0].) Combining this fact with the inversion symmetry, we get the following result, also stated in [S0].

**Lemma 2.6 (Insertion)** *If  $s < 1/2$  and  $t = s/(2s + 1)$  then  $\Delta_s \cap X_s^0$  and  $\Delta_t \cap X_t^0$  are similar.*



Figure 2.6:  $\Delta_s$  for  $s = 5/13$ .



Figure 2.7:  $\Delta_t$  for  $t = 5/23$ .

The reason for the name of this result is that the picture for  $t$  is obtained from the picture for  $s$  just by inserting 2 new central squares. The reader can see this in action by comparing Figures 2.6 and 2.7.

In view of the Insertion Lemma, we will often describe our results for  $s \in [1/4, 1)$ . The one other advantage of the Insertion Lemma, which we exploited in [S0] is that often a statement for parameters in  $[1/4, 1)$  involves a finite computational proof whereas the statement for all parameters in  $(0, 1)$  would require (if computed directly) an infinite computation.

## 2.7 Bilateral Symmetry

The kind of symmetry described in this section looks obvious from the pictures, but it required a nontrivial computational proof in [S0]. We will describe the symmetry for  $s \in [1/4, 1)$ .

We say that a line  $L$  is a *line of symmetry* for  $\Delta_s$  if

$$\Delta_s \cap (X_s \cap R_L(X_s)) \quad (5)$$

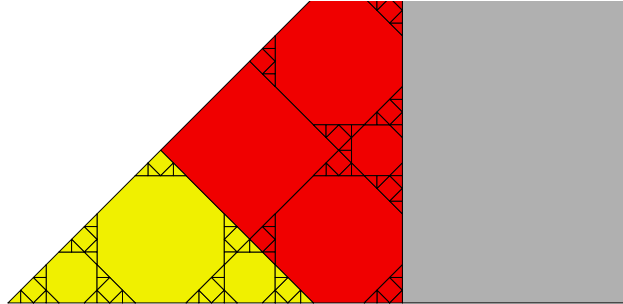
is invariant under the reflection  $R_L$  in  $L$ . Note that  $X_s$  itself need not be invariant under  $R_L$ . Consider the following lines.

- Let  $H$  be the line  $y = 0$ .
- Let  $V$  be the line  $x = -1$ .
- For  $D_s$  be the line of slope  $-1$  through  $(-s, -s)$ .
- For  $s \in [1/4, 1/2]$ ,  $E_s$  be the line of slope  $-1$  through  $(-3s, -3s)$ .
- For  $s \in [1/2, 1]$ ,  $E_s$  is the line of slope  $1$  through  $(-s, -s)$ .

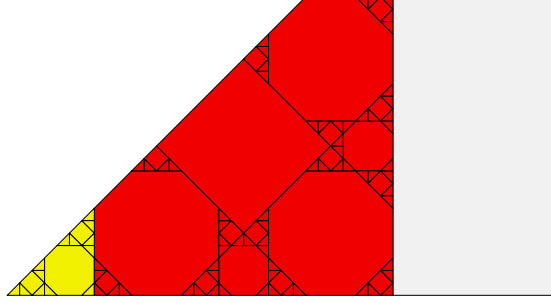
We call these the *fundamental lines of symmetry*. For each line  $L$  above, the line  $\iota(L)$  is also a line of symmetry. However, we will not usually refer to these other lines.

$H$ ,  $V$ , and  $D_s$  and  $E_s$  respectively are the lines of symmetry for

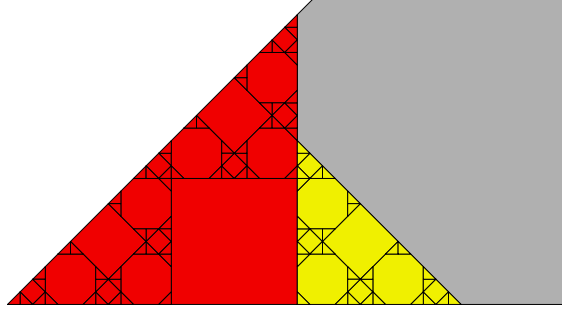
$$\begin{aligned} A_s &= (X_s \cap R_H(X_s))^0, & B_s &= X_s \cap R_V(X_s), \\ P_s &= X_s \cap R_D(X_s), & Q_s &= X_s \cap R_{E_s}(X_s). \end{aligned} \quad (6)$$



**Figure 2.8:**  $A_s$  (red) and  $B_s$  (yellow) for  $s = 12/31$ .



**Figure 2.9:**  $P_s$  (red) and  $Q_s$  (yellow) for  $s = 12/31$ .



**Figure 2.10:**  $P_s$  (red) and  $Q_s$  (yellow) for  $s = 18/23$ .

**Remarks:**

- (i) In case  $s \in (1/2, 1)$ , the bilateral symmetry behaves nicely with respect to the inversion symmetry. The value  $r = 1/(2s)$  also lies in  $(1/2, 1)$ . There is a similarity carrying  $\Delta_r$  to  $\Delta_s$ , and this similarity carries the regions  $(A_r, B_r, P_r, Q_r)$  to the regions  $(Q_s, P_s, B_s, A_s)$ . So, for  $s \in (1/2, 1)$ , the symmetries associated to  $(P_s, Q_s)$  are equivalent to the ones associated to  $(A_s, B_s)$ . This is not true for  $s < 1/2$ .
- (ii) Looking at the figures, the reader will probably be able to find other regions of bilateral symmetry. I think that all the bilateral symmetry one sees in the picture is a consequence of the ones already mentioned, together with renormalization.
- (iii) In [S0] we defined  $A_s$  to be the hexagon  $X_s \cap R_H(X_s)$ , but here it seems better just to take the portion of this hexagon which lies to the left of the central tiles of  $\Delta_s$ . This notation change causes no troubles.
- (iv) The sets  $A_s, B_s, P_s, Q_s$ , which we call *symmetric sets*, are all clean. This follows from the fact that each side of one of these sets either lies in  $\partial X_s$  or can be moved to  $\partial X_s$  by the bilateral symmetry.

## 2.8 Squares in the Tiling

Here we sketch the proof of a result from [S0] which illustrates some of the power of the results mentioned above.

**Lemma 2.7** *When  $s$  is rational and oddly even,  $\Delta_s$  contains only squares and right-angled isosceles triangles. When  $s$  is irrational and oddly even,  $\Delta_s$  contains only squares.*

**Proof:** When  $s = 1/2$  we check that  $\Delta_s$  consists entirely of squares and right-angled isosceles triangles. From the Insertion Lemma, the same result holds for  $s = 1/(2n)$  for  $n = 2, 3, 4, \dots$ . In general, the result follows from induction on the length of the orbit  $R^n(x)$  and Theorem 2.5. In [S0] we showed that the triangles vanish in the irrational limit. So, when  $s$  is irrational and oddly even,  $\Delta_s$  consists only of squares. ♠

$\Delta_s$  contains two kinds of squares. We say that a *box* is a square whose sides are parallel to the coordinate axes. We say that a *diamond* is a square whose sides have slope  $\pm 1$ . Then  $\Delta_s$  consists entirely of boxes and diamonds. This follows from induction and the same kind of proof given in Lemma 2.7.



## 3 The Pattern of Filling

### 3.1 The First Half

In this chapter we elaborate on Theorem 2.5, explaining more precisely how  $\Delta_s$  is composed of parts of  $\phi_s(\Delta_t \cap Y_t)$ . Here  $t = R(s)$  and  $\phi_s$  are as in Theorem 2.5.

We first consider the case when  $s < 1/2$ . Let  $t = R(s)$  and  $u = R(t)$ .

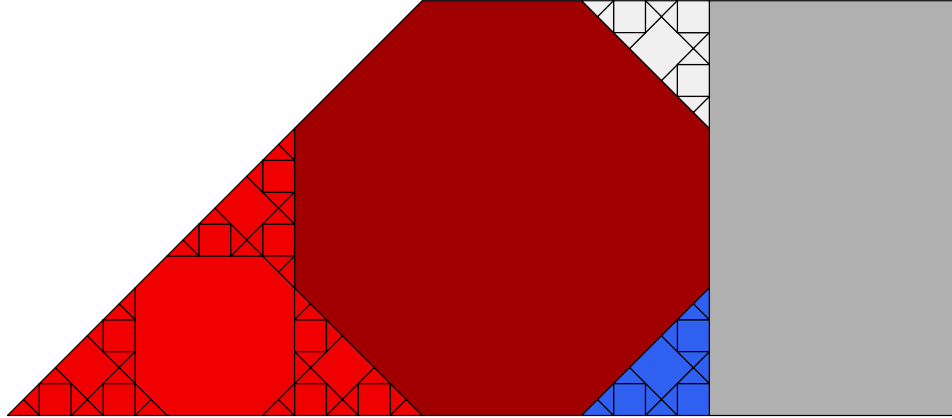
- If  $t > 1/2$  let  $K = 1$ .
- If  $t < 1/2$  and  $u < 1/2$ , let  $K = \text{floor}(1/(2t))$ .
- If  $t < 1/2$  and  $u > 1/2$ , let  $K = 1 + \text{floor}(1/(2t))$ .

The number  $K = K(s)$  plays an important role in our constructions. We call it the *layering constant*

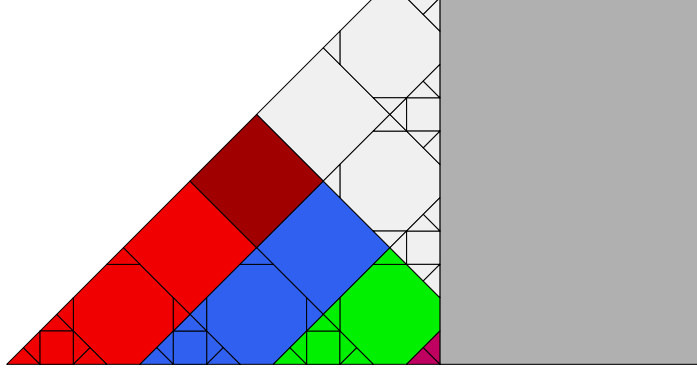
Let  $U_s$  be the image, under of  $\phi_s$ , of the trivial tile in  $\Delta_t$ . This is the dark red tile in Figures 3.1 and 3.2 below. Define

$$\Psi_s^0 = Z_s^0 \cup U_s. \quad (7)$$

Let  $\tau_s$  denote the subset of  $Z_s^0$  lying beneath the line extending the top right edge of  $U_s$ . Here  $\tau_s$  is the colored region in Figures 3.1 and 3.2.



**Figure 3.1:**  $\Psi_s^0$  (red) and  $U_s$  (dark red) and  $\tau_s$  (red, blue) for  $s = 13/44$ .



**Figure 3.2:**  $\Psi_s^j$  for  $j = 0, 1, 2, 3$  (red, blue, green, magenta) for  $s = 11/26$ .

Let  $T_s$  denote the transformation which translates by a vector pointing in the positive  $x$  direction and having length equal to the length of the bottom side of  $\Psi_s^0$ . Define

$$\Psi_s^j = T_j(\Psi_s^0) \cap \tau_s. \quad (8)$$

We have

$$\tau_s = \bigcup_{j=0}^K \Psi_s^j. \quad (9)$$

For larger  $j$ , the sets in Equation 9 are empty.

The whole tiling is determined by the tiling inside  $\tau_s$  and symmetry.

$$X_s = X_s^0 \cup \text{central tiles} \cup \iota(X_s^0), \quad X_s^0 = \tau_s \cup R_D(\tau_s). \quad (10)$$

The second equation is a consequence of the fact that the top right boundary of  $\tau_s$  is parallel to the fundamental symmetry line  $D_s$  and lies above it.

The following result explains the structure of  $\Delta_s$  inside  $\tau_s$ .

**Lemma 3.1 (Filling)**  $T_s^{-j}(\Delta_s \cap \Psi_s^j) = \Delta_s \cap T_s^{-j}(\Psi_s^j)$  for all  $j = 1, \dots, K$ .

**Proof:** What the result means is that the map  $T_s^{-j}$  respects the tiling, at least on the relevant domain.

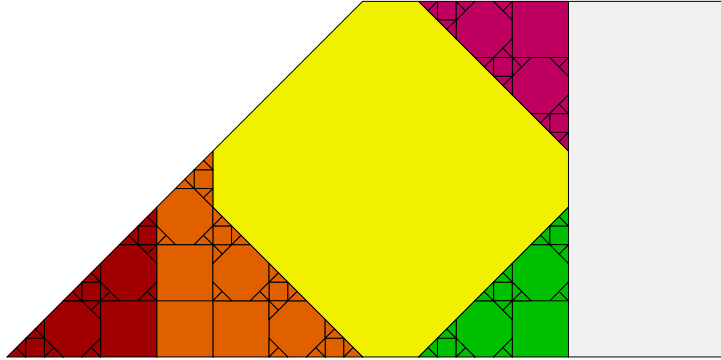
By the Insertion Lemma, we can take  $s \in (1/4, 1/2)$ . Referring to the basic map of our system, the map  $f_s : X_s \rightarrow X_s$  is defined in terms of a partition of  $X_s$ . On each piece of this partition,  $f_s$  is a translation. Let  $\Omega_s$  be the piece of the partition which shares the lower left vertex of  $X_s^0$ . A routine calculation shows that the restriction of  $f_s$  to  $\Omega_s$  is  $T_s$ .

When  $s \in (1/3, 1/2)$ , the set  $\Omega_s$  is the quadrilateral with the following properties.

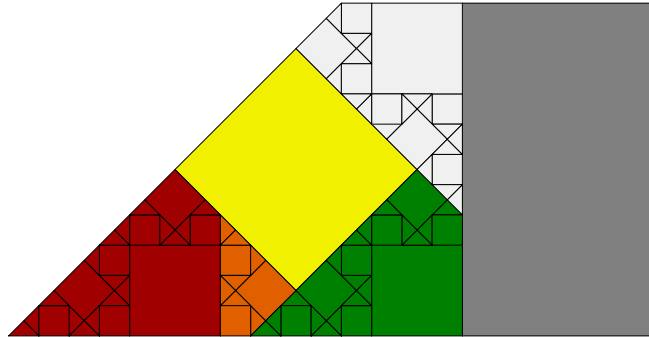
- The left edge of  $\Omega_s$  is contained in the left side of  $X_s$ .
- The bottom edge of  $\Omega_s$  is contained in the bottom edge of  $X_s$ .
- The top edge of  $\Omega_s$  lies in the same line as the top edge of  $Z_s^0$ .
- The right edge  $e_s$  of  $\Omega_s$  is vertical and has the property that  $T_s(e)$  lies in the left edge of the leftmost central tile of  $\Delta_s$ .

When  $s \in (1/4, 1/3)$ , the set  $\Omega_s$  is a triangle whose left, right, and bottom sides are as above. Our argument works the same in either case.

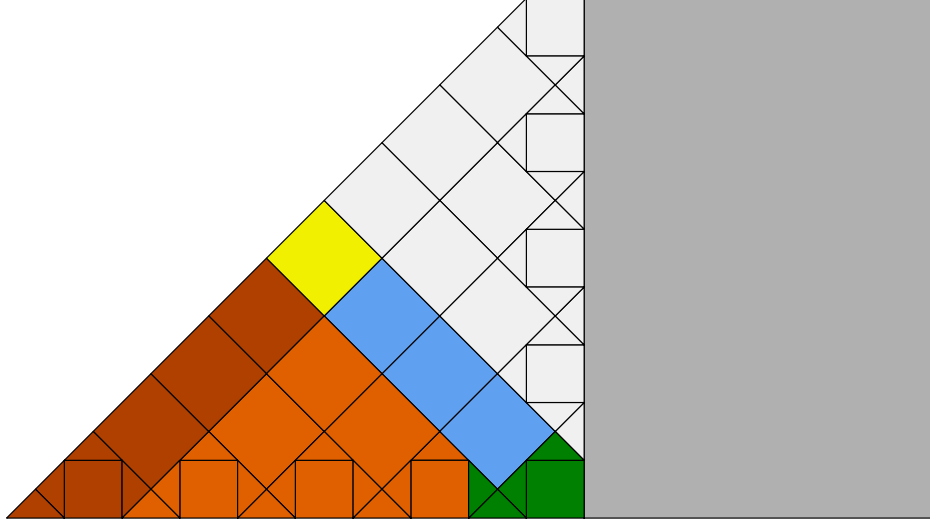
Figures 4.3-4.5 illustrate the picture for  $s = 19/60, 11/30, 9/20$ . The diagonal line  $L_s$  bisects the yellow square and the green diamond in each picture.



**Figure 3.3:**  $Z_s^0$  (red, orange),  $\Omega_s$  (red) and  $f_s(\Omega_s)$  green



**Figure 3.4:**  $Z_s^0$  (red, orange) and  $\Omega_s$  (red) and  $f_s(\Omega_s)$  (green).



**Figure 3.5:**  $Z_s^0$  (red),  $\Omega_s$  (red, orange), and  $f_s(\Omega_s)$  (green, orange, blue).

A routine calculation shows that

$$T_s^{-1}(\tau_s - \Psi_s^0) \subset \Omega_s. \quad (11)$$

If we start with any point  $p \in \tau_s - \Psi_s^0 = Z_s^0 \cup U_s$ , we see from the shape of  $\Omega_s^0$  that the iterates  $f_s^{-j}(p)$  are defined and lie in  $\Omega_s^0$ , for each  $j = 0, 1, 2, \dots$  until we reach some  $k \leq K$  such that  $f_s^{-k}(p) \subset Z_s^0$ . Our result follows from this observation and from the fact that  $\Delta_s$  is  $f_s$ -invariant. ♠

Say that a *central tile* of  $\Delta_s \cap \Psi_s^0$  is the image of a central tile of  $\Delta_t$  under the map  $\phi_s$ . For instance, in Figure 3.5, there are 4 central tiles, one yellow and 3 dark red. The central tiles all have the same size, and lie in  $\Psi_s^0$ .

For  $j < K$ , the tiling  $\Delta_s \cap \Psi_s^j$  has a simple description.

- Start with the tiling  $\Delta_s \cap \Psi_s^0$ .
- Chop off the top  $j$  central tiles.
- Translate by  $T_s^j$ .

The tiling  $\Delta_s \cap \Psi_s^K$  is more complicated, but we do not need to know it explicitly.

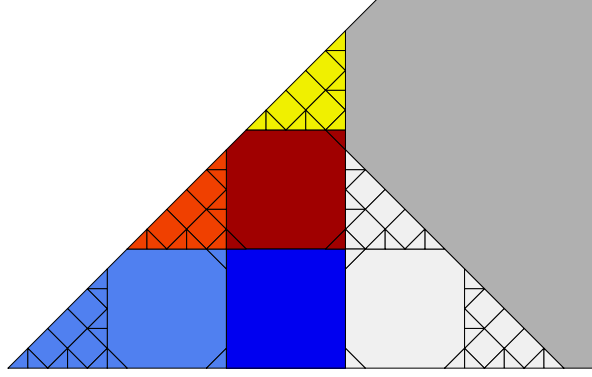
### 3.2 The Second Half

Now we explain the picture for  $s \in (1/2, 1)$ . This time, we define

$$K = \text{floor}\left(\frac{1}{2 - 2s}\right) \quad (12)$$

Again, we call  $K$  the layering constant.

This time, the right edge of  $Z_s^0$  lies on the same line as the left edge of the central tile of  $\Delta_s$ . Let  $\delta_s$  denote this line. Let  $T_s$  denote the translation by the vector which is positive proportional to  $(1, 1)$  and whose length is the same as the length of the left side of  $Z_s^0$ . Let  $\tau_s$  be the region of  $X_s^0$  lying to the left of  $\delta_s$ . In Figure 3.6, the region  $\tau_s$  is colored blue/red/yellow.



**Figure 3.6:**  $\Psi_s^j$  for  $j = 1, 2, 3$  (red, blue, yellow) for  $s = 14/17$ .

Define

$$\Psi_s^j = T_s^j(Z_s) \cap \tau_s. \quad (13)$$

We have

$$\tau_s = \bigcup_{j=0}^K \Psi_s^j. \quad (14)$$

This time we have

$$X_s = X_s^0 \cup \text{central tiles} \cup \iota(X_s^0), \quad X_s^0 \subset \tau_s \cup R_V(\tau_s), \quad (15)$$

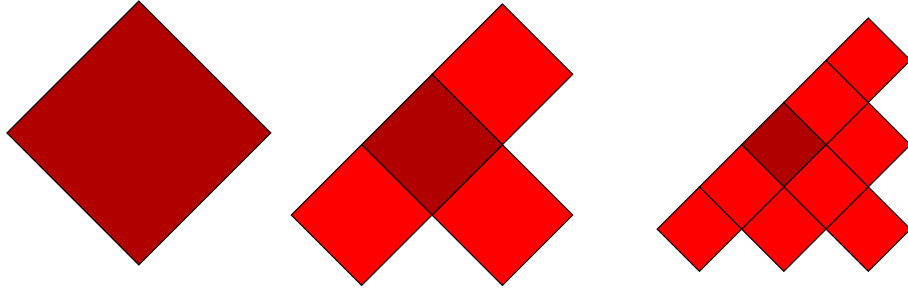
where  $V$  is the vertical line of symmetry,  $x = -1$ .

With these definitions, the Filling Lemma holds *verbatim*, and the proof is essentially the same. This time  $\Omega_s$  is a right isosceles triangle, and the left edge of  $f_s(\Omega_s)$  lies in  $\delta_s$ , and  $f_s$  translates diagonally along the vector that generates the left side of  $Z_s$ . In Figure 3.6,  $\Omega_s$  is the union of the light red and light blue tiles.

### 3.3 Pyramids

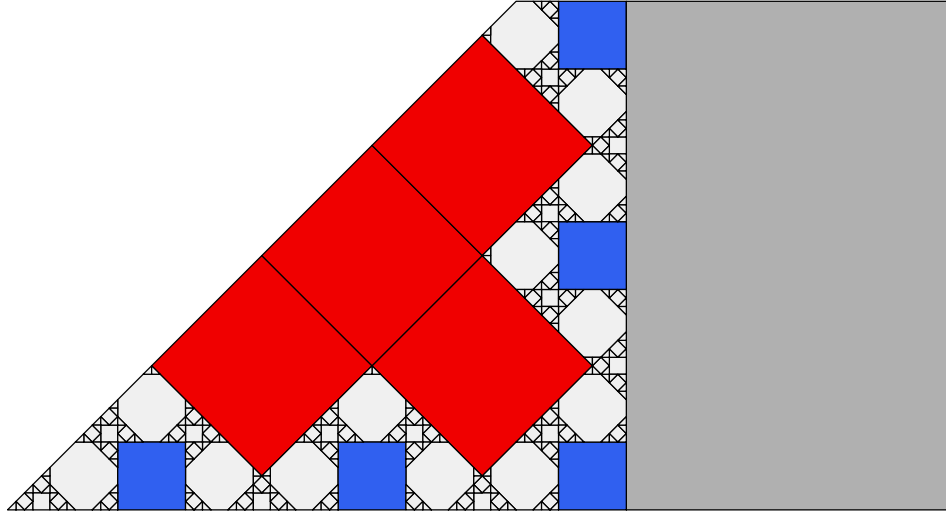
Now we explain more of the structure. We will concentrate on the case  $s < 1/2$ .

Let  $K = K(s)$  be the layering constant. Say that a *pyramid* of size  $k$  is a configuration of diamonds having the structure indicated in Figure 3.7 for  $k = 1, 2, 3$ . We refer to the longest (diagonal) row of diamonds as the *base* of the pyramid.



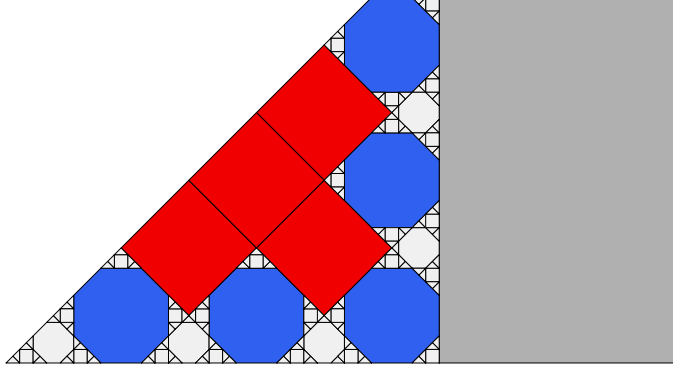
**Figure 3.7:** Pyramids for  $k = 1, 2, 3$ .

Figure 3.8 shows a red pyramid of size 2 contained in  $\Delta_s$  for  $s = 30/73$ . In the next section, we will explain the meaning of the blue squares shown in the figure.



**Figure 3.8:** The pyramid.  $s = 30/73$ ,  $R(s) = 13/60$  and  $R^2(s) = 4/13$ .

Say that an *extended pyramid* of size  $K$  is the union of polygons obtained by taking the outer squares in each row and chopping off the corners so as to leave semi-regular octagons. Figure 3.9 shows an example of an extended pyramid of size  $K$ .



**Figure 3.9:** Extended Pyramid.  $s = 27/64$ ,  $R(s) = 5/27$  and  $R^2(s) = 7/10$ .

Say that  $s < 1/2$  has *type 0* if  $R(s) < 1/2$  and  $R^2(s) < 1/2$ . Otherwise, say that  $s$  has *type 1*.

**Lemma 3.2** *Suppose  $s$  has type 0 (respectively type 1). Then  $\Delta_s$  contains an ordinary (respectively extended) pyramid of size  $K$ .*

**Proof:** Suppose first that  $R^2(s) < 1/2$ . The bottom  $K$  squares in the base of the desired pyramid are guaranteed by Theorem 2.5 and the Insertion Lemma (applied to  $\Delta_t$ ). The bottom half of the pyramid is then guaranteed by the Filling Lemma. The top half is then guaranteed by the bilateral symmetry corresponding to the diagonal line  $D_s$ .

The proof is essentially the same when  $R^2(s) > 1/2$ . What happens here is that the tiles in  $\Delta_t$  just to the left of the leftmost central tile is an octagon which has the same height as the central tiles. have the same width as the central tiles. Here  $t = R(s)$ . If we simply keep track of this additional octagon, we get the same structure as in the other case, except that these octagons replace the outer layer of diamonds. ♠

**Remark:** One has a result similar to Lemma 3.2 when  $s > 1/2$ . When  $s > 1/2$  and  $R^2(s) < 1/2$  we get an ordinary pyramid of size  $K - 1$ . When  $s > 1/2$  and  $R^2(s) > 1/2$  we get an extended pyramid of size  $K$ .

### 3.4 The Octagrid

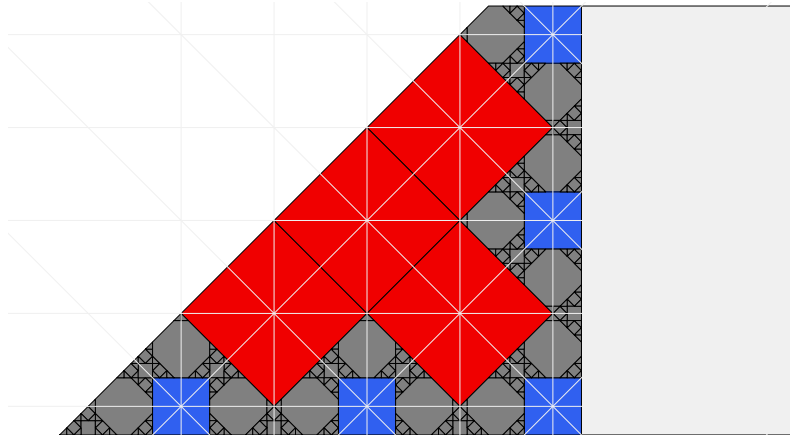
**Remark:** The reader only interested in Statement 1 of the Main Theorem can skip the rest of this chapter.

In this section, we consider the case when  $s < 1/2$  and  $R(s) < 1/2$ . One can do something similar for other parameter ranges, but we do not need the construction otherwise.

First we consider the case  $R^2(s) > 1/2$ . Thanks to the equalities between the widths, the distance between the center of an octagon and the center of an adjacent square in the extended pyramid is the same as the distance between two adjacent squares. Put another way, the union of the centers of the tiles in the extended pyramid is contained in a square grid.

When  $R^2(s) < 1/2$ , there is a similar phenomenon. This is where the blue squares in Figure 3.8 (and Figure 3.10) come in. These squares have the following description. The bottom left blue square is the image, under  $\phi_s$ , of the central tile of  $Z_t$ . The remaining blue tiles exist as a consequence of the Filling Lemma and bilateral symmetry. We call the union of the pyramid and these blue squares the *extended pyramid*. The union of the centers of the tiles of the extended pyramid, in this case also, is part of a square grid. This follows from a routine calculation, which we omit.

In either case, we form the *octagrid* by taking the union of horizontal, vertical, and diagonal (meaning slope= $\pm 1$ ) lines through the centers of the tiles of the extended pyramid. The octagrid chops up  $\Delta_s$  into what turn out to be usefully small pieces.



**Figure 3.10:** The octagrid for  $s = 30/73$ .



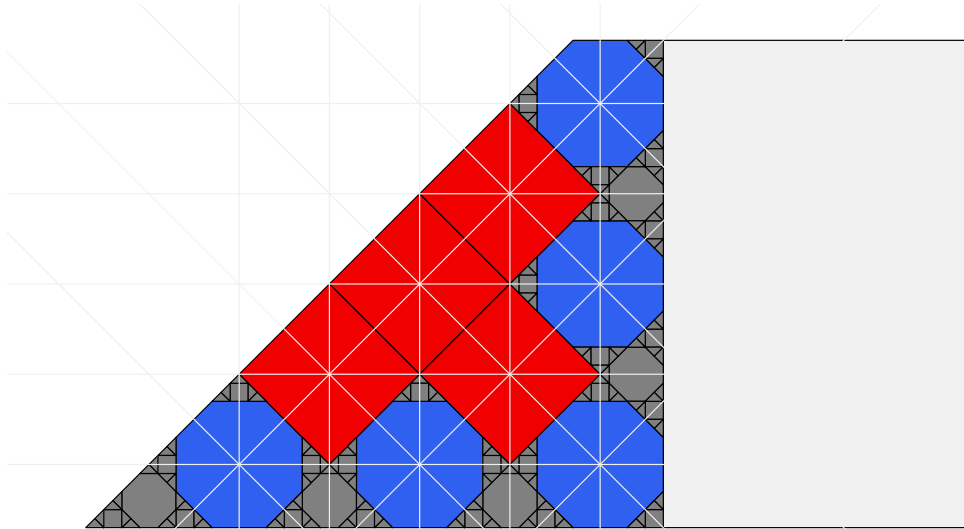
There are finitely many lines in the octagrid. These lines partition  $X_s^0$  into finitely many bounded convex regions. We call these regions *octagrid components*. Each octagrid component is some open convex polygon.

**Lemma 3.3** *Let  $G$  be any octagrid component. There is an isometry carrying  $\Delta_s \cap G$  to a subset of  $\Delta_s \cap Z_s^0$ .*

**Proof:** The fundamental line  $D_s$  of symmetry is contained in the octagrid. The reflection  $R_d$  carries each octagrid component above  $D$  into some octagrid component below  $D$ . Hence, by symmetry, it suffices to prove our result when  $G$  lies beneath  $D_s$ .

$G$  lies in the region  $\tau_s$  from the Filling Lemma. Applying the map  $T_s^{-1}$  from Equation 8 as many times as we can, we can assume that (the interior of)  $G$  intersects  $Z_s^0$ . If  $G \subset Z_s^0$ , we are finished. If  $G$  is contained in one of the squares of the pyramid, we are finished.

The only remaining possibility is that  $G$  has a vertex on the centerline of  $Z_s^0$  (the line  $\phi_s(H)$ ) but extends over the right edge of  $Z_s^0$ . The problem is that the diagonal edge of  $G$  is perpendicular rather than parallel to the centerline of  $Z_s^0$ . To fix this problem, we consider  $G' = T_s^j \circ R_V(G)$  for the largest value of  $j$  we can take. The interior of  $G'$  intersects  $Z_s^0$  and the diagonal edge of  $G'$  is parallel to  $Z_s^0$ . Hence,  $G' \subset Z_s^0$ . Since we have only applied symmetries of the tiling, we see that  $\Delta_s \cap G$  and  $\Delta_s \cap G'$  are isometric. ♠



**Figure 3.11:** The octagrid for  $s = 27/64$ .

## 4 Geometric Lemmas

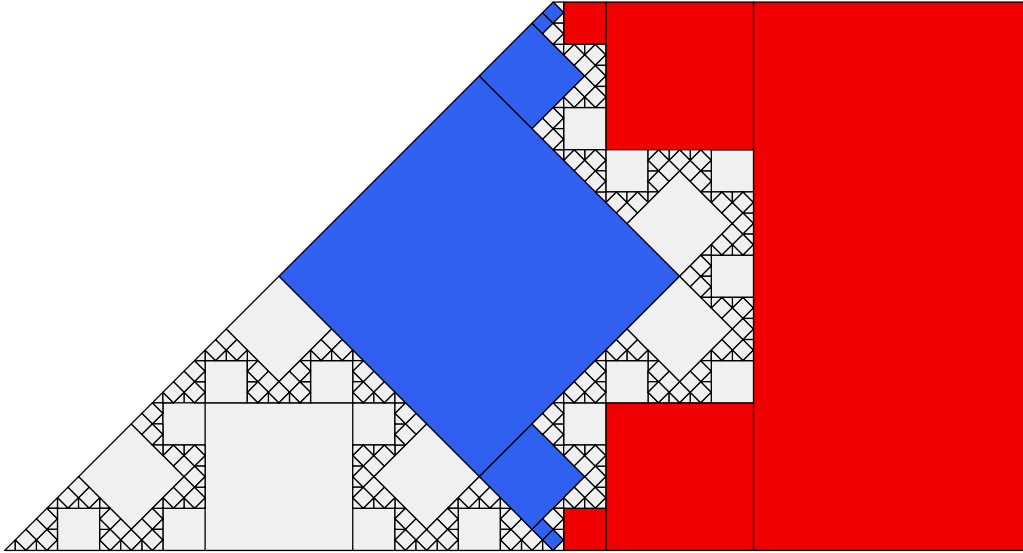
In this chapter we prove 3 geometric lemmas, which we call the Shield Lemma, the Pinching Lemma, and the Covering Lemma. Only Statement 3 of the Main Theorem requires the Covering Lemma.

### 4.1 The Shield Lemma

Let  $A_s$  be the symmetric piece from §2.7.

**Lemma 4.1 (Shield)** *Let  $s$  be irrational and oddly even. Every point of  $\partial A_s$ , except the 2 vertices having obtuse angles, is contained in the edge of square. Those points which belong to the boundaries of more than one tile are the vertices of pairs of adjacent squares.*

Figure 4.1 illustrates the Shield Lemma. When  $s$  is rational, there are 2 small triangles touching the obtuse vertices of  $A_s$ . These triangles vanish in the irrational limit.



**Figure 4.1:**  $A_s$  for  $s = 26/71$ .

We define the *shield*  $\Sigma_s$  to be the union of the top left edge of  $A_s$  and the left half of the top edge.  $\Sigma_s$  is the union of two line segments. The top left vertex  $\nu_s$  of  $X_s$  (and  $A_s$ ) is the place where the two line segments join.

By symmetry, it suffices to prove the Shield Lemma for the points of  $\partial A_s$  contained in the shield. We analyze the picture in the rational case and then take limits. In this section we work out how  $\Delta_s$  sits in  $A_s$  when  $s$  is rational and oddly even.

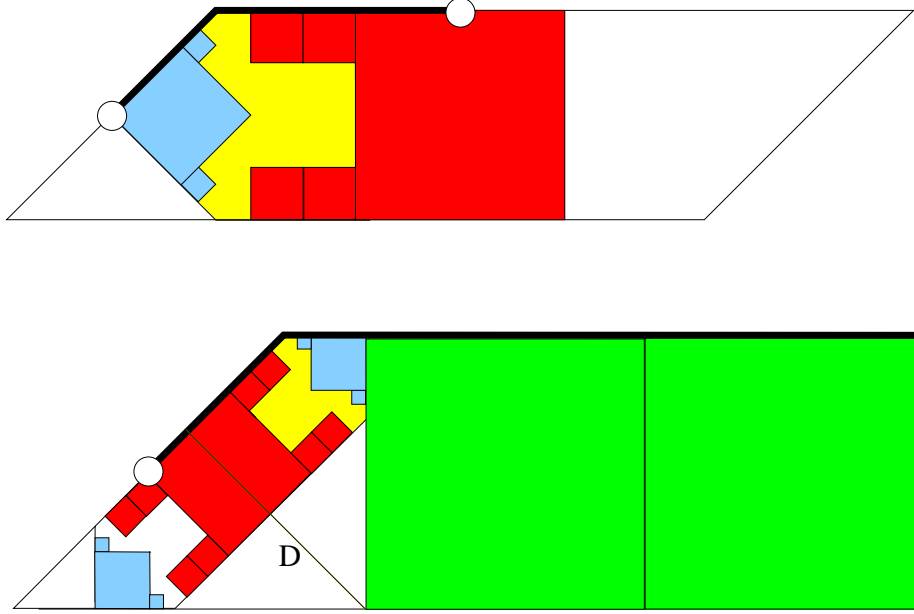
We say that a square of  $\Delta_s$  *abuts*  $\Sigma_s$  if  $\overline{T} \cap \Sigma_s$  is nonempty. In this case, the intersection is a line segment which we call the *contact* between the square and the shield. The *radius* of  $T$  is the distance from the center of  $T$  to a corner of  $T$ . We call a radius  $\rho$  *realized*, if a square of  $\Delta_s$  having radius  $\rho$  abuts the shield.

**Lemma 4.2** *Let  $s$  be an oddly even rational. The following is true.*

1. *Exactly one triangle of  $\Delta_s$  abuts  $\Sigma_s$  and the segment of contact contains  $\nu_s$  as an endpoint.*
2. *The squares which abut  $\Sigma_s$  occur in monotone decreasing size, largest to smallest, as one moves from an endpoint of  $\Sigma_s$  to  $\nu_s$ .*
3. *The number of squares of each size is determined by the even expansion of  $s$ .*
4. *Let  $\rho$  be a realized radius. Some square of radius  $\rho$ , which abuts  $\Sigma_s$ , has a vertex within  $\rho$  of  $\nu_s$ .*

Let us assume Lemma 4.2 for now, and finish the proof of the Shield Lemma. Let  $\{r_n\}$  be a sequence of oddly even rationals which converges to  $s$ . Given the convergence of tilings described above, we see that the union of square tiles abutting  $\Sigma_s$  is the Hausdorff limit of the union of square tiles abutting  $\Sigma_{r_n}$ , as  $n \rightarrow \infty$ . The size of the single triangle in the picture for the rational parameters tends to 0. Hence, every point of  $\Sigma_s - \nu_s$  is contained in a segment of contact for some square. The main point to worry about is that somehow there is a point  $p \in \Sigma_s - \nu_s$ , with the following property: As  $n$  tends to  $\infty$ , the square whose segment of contact contains  $p$  tends to 0 in size. This unfortunate situation cannot occur because it would violate Item 4 of Lemma 4.2.

**Proof of Lemma 4.2:** The proof goes by induction on the length of the orbit  $\{R^n(s)\}$ . When  $s = 1/2$  the result holds by inspection. The case  $s = 1/2n$  follows from the Insertion Lemma. For  $s \neq 1/2n$ , let  $t = R(s)$ . By induction, all the properties of the lemma hold for  $\Delta_t$ .



**Figure 4.2:** Inherited structure

Let  $D = D_s$  be the diagonal line of bilateral symmetry and let  $R_D$  denote reflection in  $D$ . The yellow region at the top left of Figure 4.2 is  $A_t$ . The big blue square in the top of Figure 4.2, which is the central square of  $\Psi_t^0$ , is mapped by  $R_D \circ \phi_s$  to a square which abuts the (green) leftmost central square of  $\Delta_s$ .

The pattern of tiles abutting the shield  $\Sigma_s$  is the same as the pattern of squares abutting the shield  $\Sigma_t$ , except that some green ones have been appended. The number of green ones is determined by the number  $n_0$  in the even expansion of  $s$ . All the points in our lemma follow from this structure. ♠

**Corollary 4.3** *Let  $S_s$  denote the left half of  $\hat{\Lambda}_s$ . There exists a convex set  $D_2$  such that  $S_s \cap \text{interior}(A_s) \subset D_2$  and  $D_2$  intersects  $\partial A_s$  only at the two obtuse vertices.*

**Proof:** We simply slice off from  $A_s$  suitably chosen neighborhoods of the square tiles which abut the edges of  $A_s$ . With a little care (i.e., by making these neighborhoods shrink very rapidly as we approach the vertices) we can make the resulting set convex. ♠

## 4.2 The Pinching Lemma

Let  $S_s$  denote the left half of the limit set  $\widehat{\Lambda}_s$ .

**Lemma 4.4 (Pinching)** *At most one point of  $S_s$  lies on each fundamental line of bilateral symmetry.*

We argue by contradiction. Let  $g(s)$  denote the diameter of  $X_s^0$ . Say that a *counterexample* is a quadruple  $\Omega = (L, p_1, p_2, s)$ , where  $L = L_s$  is a fundamental line of symmetry and  $p_1 \neq p_2 \in L \cap S_s$ . We call  $s$  the *parameter* of the counterexample. We define

$$\lambda(\Omega) = \frac{\|p_1 - p_2\|}{g(s)}. \quad (16)$$

If  $\Omega'$  is another counterexample obtained from  $\Omega$  using either the Insertion Lemma or (when  $s > 1/2$ ) the Inversion Lemma, we have  $\lambda(\Omega) = \lambda(\Omega')$ . Moreover

$$\lambda = \sup_{\Omega} \lambda(\Omega) \leq 1 \quad (17)$$

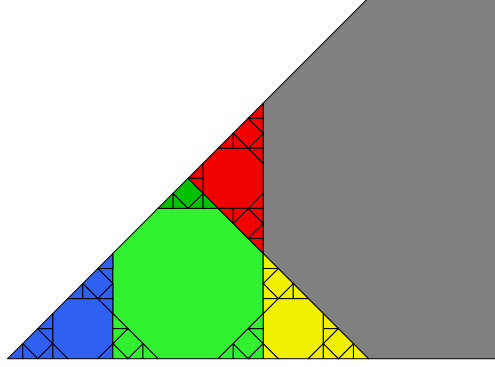
**Lemma 4.5** *For any  $\epsilon > 0$  there exists a counterexample  $\Omega$ , whose parameter is less than  $1/2$ , such that  $\lambda(\Omega) > \lambda - \epsilon$ .*

**Proof:** Certainly, there exists a counterexample  $\Omega = (L, p_1, p_2, s)$  such that  $\lambda(\Omega) > \lambda - \epsilon$ . If  $s < 1/2$  we are finished. If  $s > \sqrt{2}/2$  we can apply the Inversion Lemma to reduce to the case  $s \in (1/2, \sqrt{2}/2]$ . In this case, we have  $K = 1$ , where  $K$  is the layering constant.

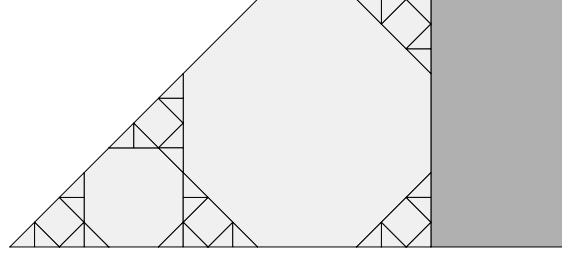
Let  $t = R(s) = 1 - s$ . When  $K = 1$  we have the following facts.

- $R_D \circ \phi_s(B_t) = A_s$  (red) is a neighborhood of  $H$  in  $A_s$ .
- $R_D \circ \phi_s(A_t)$  (green) is a neighborhood of  $V$  in  $B_s$ .
- $\phi_s(P_t)$  (light green) is a neighborhood of  $D_s$  in  $P_s$ .
- $R_V \circ \phi_s(Q_t)$  (yellow) is a neighborhood of  $E_s$  in  $Q_s$ .

The colors refer to the various regions in  $\Delta_s$  shown in Figure 4.3. In all cases, we can use the relevant map to pull back the counterexample isomerically to  $\Delta_t$ , and we get a counterexample involving  $t$  with the same diameter. But then the  $\lambda$ -value of this counterexample does not decrease. Hence, the new counterexample  $\Omega'$  satisfies  $\lambda(\Omega') > \lambda - \epsilon$  and has parameter  $t < 1/2$ . ♠



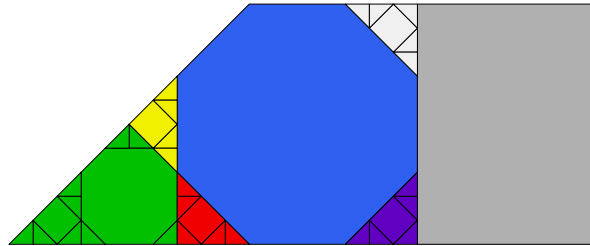
**Figure 4.3:** Various regions in  $\Delta_s$  for  $s = 12/17$ .



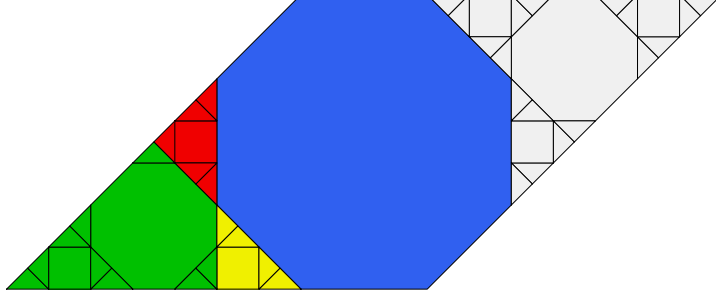
**Figure 4.4:** The left half of  $\Delta_t$  for  $t = R(s) = 5/17$ .

By the previous result, and the Insertion Lemma, we can find a counterexample  $\Omega$  having parameter  $s \in (1/4, 1/2)$  such that  $\lambda(\Omega)$  is as close as we like to  $\lambda$ .

Consider a counterexample  $\Omega$  having  $s \in (1/4, 1/3)$ . For  $s$  in this range, we have  $t = R(s) > 1/2$ . The blue octagon in  $\Delta_s$  is the image under  $\phi_s$ , of the trivial tile in  $\Delta_t$ . Call this octagon  $O_s$ .



**Figure 4.5:**  $O_s$  (blue) and other regions for  $s = 5/17$ .



**Figure 4.6:**  $\Delta_t$  for  $t = R(5/17) = 7/10$ .

We have the following facts.

- $\phi_s(Q_t)$  (yellow) is a neighborhood of  $H^0 - O_s$ .
- $\phi_s(P_t) = B_s$  (red, green).
- $R_H \circ R_D \circ \phi_s(A_t)$  (purple) is a neighborhood of  $D_s - O_s$ .
- $\phi_s(B_t) = Q_s$  (green, yellow).

The colors refer to Figures 5.4 and 5.5. In all cases, we can pull the counterexample back by the relevant similarity to get a counterexample associated to the parameter  $t$ . Call the new counterexample  $\Omega'$ . Since  $Y_t^0 = X_t^0$  for  $t > 1/2$ , we have

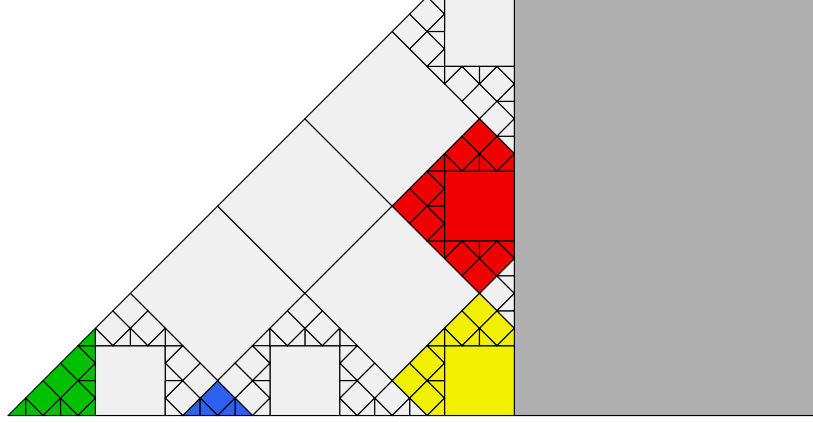
$$\begin{aligned} \lambda(\Omega') &= \frac{\|\phi_s^{-1}(p) - \phi_s^{-1}(q)\|}{\text{diam}(Y_t^0)} = \frac{\|p - q\|}{\text{diam}(\phi_s(X_t^0))} = \\ &= \frac{\|p - q\|}{\text{diam}(Z_s^0)} = \lambda(\Omega) \frac{\text{diam}(X_s^0)}{\text{diam}(Z_s^0)} = K_s \lambda(\Omega). \end{aligned} \quad (18)$$

Here the constant  $K_s > 1$  is uniformly bounded away from 1, by compactness. But this is a contradiction. We could choose  $\lambda(\Omega) > \lambda/K_s$  and then we would have  $\lambda(\Omega') > \lambda$ .

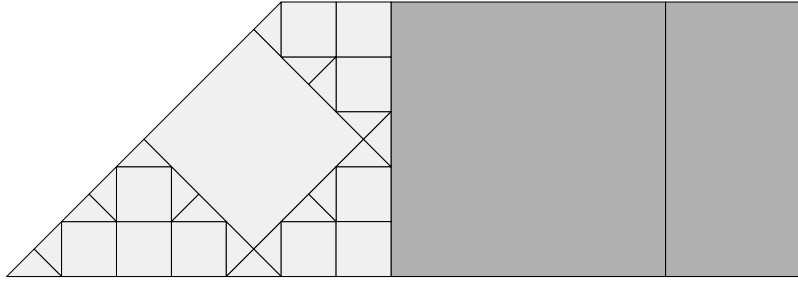
Now consider a counterexample with  $s \in (1/2, 1/3)$ . The argument is very similar to what we just did. This time, we let  $O_s$  be the pyramid from Lemma 3.2. In this case, the same remarks apply about the points of the counterexample being disjoint from (the interior of)  $O_s$ . We have the following facts.

- $R_D \circ T_s^j \circ \phi_s(P_t)$  (red) is a neighborhood of  $H^0 - O_s$  for a suitable choice of  $j$ . Here  $T_s^j$  is as in Equation 8.

- $T_s^j \circ \phi_s(Q_t)$  (blue) is a neighborhood of  $V - O_s$ .
- $R_V \circ \phi_s(A_t^0)$  (yellow) is a neighborhood of  $D - O_s$ .
- $E_s = \phi_s(B_t)$  (green).



**Figure 4.7:** Some regions of  $\Delta_s$  for  $s = 12/29$ .



**Figure 4.8:** The left half of  $\Delta_t$  for  $t = R(12/29) = 5/24$ .

This time, we get

$$\lambda(\Omega') = \lambda(\Omega) \frac{\text{diam}(X_s^0)}{\text{diam}(\phi_s(X_t^0))} = L_s \lambda(\Omega) \quad (19)$$

Here  $L_s > 1$  is uniformly bounded away from 1. We get the same contradiction as before. This completes the proof of the Pinching Lemma.



### 4.3 The Covering Lemma

We say that an  $\epsilon$ -patch is a triple  $(K, \psi, u)$  where

- $u \in (0, 1)$ .
- $K$  is one of the 4 symmetric sets  $A_u, B_u, P_u, Q_u$  from §2.7.
- $\psi : K \rightarrow X_s$  is a similarity which contracts by some factor  $\lambda \leq \epsilon$ .
- $\psi(\Delta_u \cap K) = \Delta_s \cap \psi(K)$ .

The last condition means that  $\psi$  gives a bijection between tiles of  $\Delta_u \cap K$  and tiles of  $\Delta_s \cap \psi(K)$ . When we have an  $\epsilon$ -patch  $(K, \psi, u)$ , we are recognizing a small portion of  $\Delta_s$  as being a similar copy of a large portion of  $\Delta_u$ . When the choice of  $\epsilon$  is not relevant to the discussion, we will just say *patch* in place of  $\epsilon$ -patch.

We present two versions of our result. In these versions, the constants  $m$  and  $n$  depend on everything in sight. They are meant to (possibly) vary from case to case. Here is the first version.

**Lemma 4.6** *Suppose  $s \in (0, 1)$  is irrational, and let  $t = R(s)$ . Then for each set  $K_s \in \{A_s, B_s, P_s, Q_s\}$ , we have*

$$K_s = \bigcup_{i=1}^m \alpha_i \cup \bigcup_{j=1}^n \beta_j,$$

where  $\alpha_i$  is a tile of  $\Delta_s$  and  $\beta_j$  is the image of some  $\epsilon$ -patch  $(K_j, \psi_j, t)$ . Here  $\epsilon$  is the scale factor of  $\phi_s$ , the map from Theorem 2.5.

Iterating Lemma 4.6 for the sequence  $\{R^n(x)\}$  and using the fact that  $X_s^0 = A_s \cup B_s = P_s \cup Q_s$ , we get the following corollary.

**Corollary 4.7 (Covering)** *For any  $\epsilon > 0$  we have*

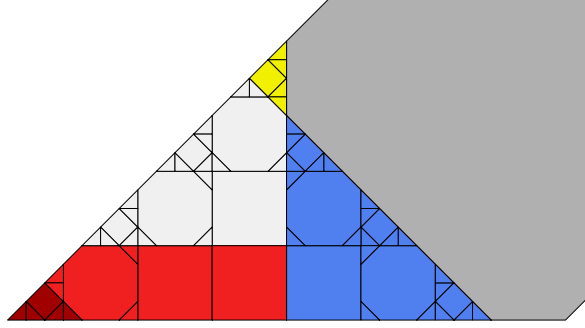
$$\Delta_s = \bigcup_{i=1}^m \alpha_i \cup \bigcup_{j=1}^n \beta_j, \tag{20}$$

where  $\alpha_i$  is a tile of  $\Delta_s$  and  $\beta_j$  is the image of some  $\epsilon$ -patch  $(K_j, \psi_j, u)$ . Moreover, we can take  $u = R^k(s)$  for any sufficiently large  $s$ . Equation 20 likewise holds for  $\Delta_s \cap K_s$  in place of  $\Delta_s$  for each symmetric piece  $K_s$ .

Lemma 4.6 has the same kind of proof as the Pinching Lemma. Let  $K = K(s)$  be the layering constant for  $s$  which appears in the Filling Lemma.

**Lemma 4.8** *Lemma 4.6 is true when  $s > 1/2$  and  $K(s) > 1$ .*

**Proof:** Figure 4.9 illustrates a typical picture for  $s$  in this range.



**Figure 4.9:** The tiling  $\Delta_s^0$  for  $s = 13/15$ . Here  $K(s) = 3$ .

We have the following equations.

- $A_s = R_D \circ \phi_s(B_t)$ .  $A_s$  is colored yellow in Figure 4.9, and  $\phi_s(B_t)$  is colored dark red.  $R_D$  is reflection in the diagonal line  $D = D_s$  of symmetry.

•

$$B_s = \nu \cup R_V(\nu) \cup R_D \circ \psi_s \circ (A_t) \cup \Theta, \quad \nu = \bigcup_{k=0}^{K-1} \Psi_s^k.$$

The set  $B_s$  is colored red/white/blue in Figure 4.9. Here  $V$  is the vertical line of symmetry, and  $\Theta$  is a finite union of square tiles.

- $P_s = \nu \cup R_D(\nu)$ . In Figure 4.9,  $P_s$  is colored red/white/yellow.

•

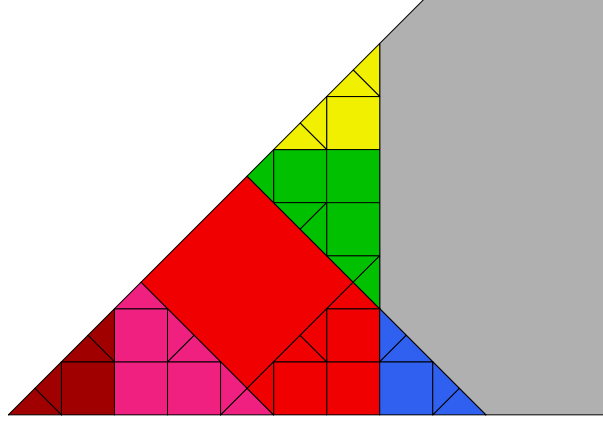
$$Q_s = R_D(\nu'), \quad \nu' = T_s^{-1} \left( \bigcup_{k=1}^{K-1} \Psi_s^k \right).$$

Here  $T_s$  is as in Equation 8 and  $Q_s$  is colored blue in Figure 4.9.

We can interpret all these equations as the desired patch coverings. ♠

**Lemma 4.9** *Lemma 4.6 is true when  $s > 1/2$ .*

**Proof:** It only remains to treat the case when  $K(s) = 1$ . Figure 4.10 shows a representative picture for  $s$  in this range.



**Figure 4.10:** The tiling  $\Delta_s^0$  for  $s = 9/14$ . Here  $K(s) = 1$ .

- $A_s = R_D \circ \phi_s(B_t)$ . (This is as above.) In Figure 4.10,  $A_s$  is colored yellow/green, and  $\phi_s(B_t)$  is colored dark red/pink.

•

$$B_s = R_d \circ \phi_s(A_t) \cup \phi_s(Q_t) \cup R_V \circ \phi_s(Q_t).$$

$B_s$  is the union of all the pieces not colored <sup>3</sup> yellow or green. The first set on the right is colored red/pink, the second set is colored dark red, and the third set is colored blue.

•

$$P_s = \phi_s(P_t) \cup \phi_s(Q_t) \cup R_D \circ \phi_s(Q_t).$$

$P_s$  is the union of all the pieces not colored blue. The first set on the right is colored pink/red/green, and the second set is colored dark red, and the third set is colored yellow.

- $Q_s = \phi_s(Q_t)$ . This set is colored dark red.

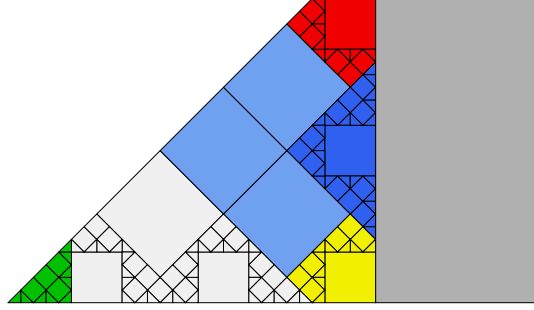
We can interpret all these equations as the desired patch coverings. ♠

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<sup>3</sup>The grey tile, which is a central tile, is always excluded.

**Lemma 4.10** *Lemma 4.6 is true when  $s < 1/2$  and  $K(s) > 1$ .*

**Proof:** Figure 4.11 shows a representative picture for  $s$  in this range.



**Figure 4.11:** The tiling  $\Delta_s^0$  for  $s = 12/29$ . Here  $K(s) = 1$ .

- $A_s = R_D \circ \phi_s(A_t) \cup R_H \circ R_D \circ \phi_s(A_t) \cup R_D(\nu') \cup R_H \circ R_D(\nu') \cup \Theta$ ,

$$\nu' = \bigcup_{k=1}^{K-1} \Psi_s^k.$$

Here  $\Theta$  is a finite union of square tiles, colored light blue. The set  $A_s$  is the union of tiles in  $X_s^0$  not colored white or green (or grey). The first set on the right is colored red. The second set is colored yellow. The union of the third and fourth sets is colored dark blue.

•

$$B_s = \nu \cup R_V(\nu) - \Theta, \quad \nu = \bigcup_{k=0}^{K-1} \Psi_s^k.$$

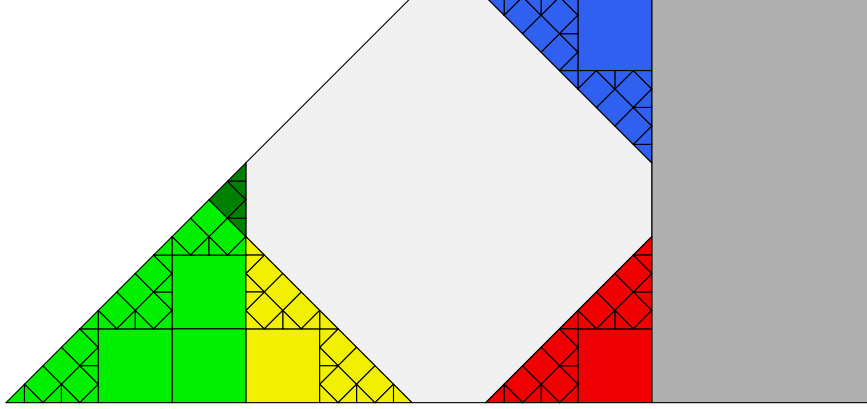
Here  $\Theta$  is a finite union of squares (colored light blue) which we delete from our union. The point is that the top square of each  $\Psi_s^k$  lies above  $B_s$ .  $B_s$  is colored white and green.

- $P_s = A_s \cup R_D(A_s)$ .
- $Q_s = \phi_s(B_t)$ .

We can interpret all these equations as the desired patch coverings. ♠

**Lemma 4.11** *Lemma 4.6 is true when  $s < 1/2$ .*

**Proof:** In view of what we have already shown, it suffices to consider the case when  $K(s) = 1$ . Figure 4.12 shows a representative picture in this case.



**Figure 4.12:** The tiling  $\Delta_s^0$  for  $s = 11/35$ . Here  $K(s) = 1$ .

•

$$A_s = R_D \circ \phi_s(A_t) \cup R_H \circ R_D \circ \phi_s(A_t) \cup \phi_s(Q_t) \cup \Theta.$$

$A_s$  is colored red/white/blue/green.  $\phi_s(A_t)$  is colored yellow. The sets on the right are respectively colored blue, red, green, and white.

- $B_s = \phi_s(P_t)$ . This set is colored yellow and light green.

•

$$P_s = \phi_s(A_t) \cup R_D \circ \phi_s(A_t) \cup R_H \circ R_D \circ \phi_s(A_t) \cup \Theta.$$

Here  $P_s$  is the union of all tiles not colored green (or grey). The sets on the right are, respectively, colored yellow, blue, red, and white.

- $Q_s = \phi_s(B_t)$ . This set is colored light and dark green.

We can interpret all these equations as the desired patch coverings. ♠

We have exhausted all the cases. This completes the proof of Lemma 4.6.

## 5 Proof of Statement 1

### 5.1 The Easy Direction

Our goal is to prove that  $\widehat{\Lambda}_s$  is a disjoint union of two arcs if and only if  $s$  is oddly even. Let  $S_s$  denote the left half of  $\widehat{\Lambda}_s$ . By symmetry, the result we want is equivalent to the statement that  $S_s$  is an arc if and only if  $s$  is oddly even.

First we prove that  $S_s$  is an arc only if  $s$  is oddly even. This is the easier of the two directions.

**Lemma 5.1** *For each integer  $k$  such that  $R^k(s) > 1/2$ , there is an octagon  $O_k$  which having one edge in the left side of  $X_s$  and one edge in the bottom side of  $X_s$ . If there are two distinct indices  $k$  and  $\ell$  with this property, then the octagons  $O_k$  and  $O_\ell$  are distinct.*

**Proof:** Say that an octagon is *wedged* in a parallelogram (of the kind we are considering) if one edge of the octagon lies in the left edge of the parallelogram and another edge lies in the bottom edge.

Let  $s_0 = s$  and  $s_k = R^k(s)$ . For ease of notation, we set  $\phi_k = \phi_{s_k}$  and  $\Delta_k = \Delta_{s_k}$ , etc. When  $s_k > 1/2$ , the central tile  $C_k$  of  $\Delta_k$  is an octagon wedged into  $X_k$ . By Theorem 2.5, the octagon  $\phi_{k-1}(C_k)$  is a tile of  $\Delta_{k-1}^0$ , and is wedged into  $X_{k-1}$ .

Iterating Theorem 2.5, we see that

$$O_k = \phi_0 \circ \dots \circ \phi_{k-1}(C_k) \tag{21}$$

is wedged into  $X_0$ .

Suppose that  $\ell > k$  is another index such that  $s_\ell > 1/2$ . Then the two octagons

$$C_k, \quad \phi_k \circ \dots \circ \phi_{\ell-1}(C_\ell)$$

are distinct because one octagon is the central tile of  $\Delta_k$  and the other one is not. But then  $O_k$  and  $O_\ell$  are the images of the above octagons under the same similarity. Hence, they are distinct. ♠

**Corollary 5.2** *If  $R^n(s) > 1/2$  for at least  $K$  different positive indices, then  $S_s$  has at least  $K + 1$  connected components.*

**Proof:** The  $K$  octagons guaranteed by Lemma 5.1 are all distinct. Call these octagons  $O_1, \dots, O_K$ . Each of these octagons is wedged into  $X_s$ , and so the union of these octagons separates  $X_s^0$  into  $K + 1$  connected components. We just need to see that  $S_s$  intersects each component.

Each octagon  $O_j$  has two vertices in the bottom edge of  $X_s$ . At each of these vertices, the adjacent edge of  $O_j$  makes an acute angle with the bottom edge of  $X_s$ . (The angle is  $\pi/4$ .) Since the  $\Delta_s$  consists of an open dense (in fact full measure) set of squares and semi-regular octagons, every neighborhood of the two vertices in question must intersect infinitely many tiles of  $\Delta_s$ . Hence, the two bottom vertices of  $O_j$  lie in  $S_s$ . This proves what we want. ♠

What we have shown is that  $S_s$  is not an arc if  $R^n(s) > 1/2$  for some  $n > 0$ .

**Lemma 5.3** *If  $s > 1/2$ , then  $S_s$  is not an arc.*

**Proof:** Let  $C$  be the trivial tile of  $\Delta_s$ . The same argument as in the previous lemma shows that  $S_s$  contains the following three points.

1. The vertex where the vertical edge of  $C$  meets the left edge of  $X_s$ . This is the top vertex of both  $A_s$  and  $P_s$ .
2. The vertex where the slope  $-1$  edge of  $C$  meets the bottom edge of  $X_s$ . This is the right vertex of  $B_s$  and of  $Q_s$ .
3. The bottom left vertex of  $X_s$ . This is the left vertex of  $B_s$  and of  $P_s$ .

Any arc connecting these 3 points, in some order, must cross at least twice one of the symmetry lines from the Pinching Lemma. Hence  $S_s$  cannot be an arc. ♠

**Remark:** In fact  $S_s$  is homeomorphic to a "Y" when  $s > 0$  and  $R^n(s) < 1/2$  for all  $n > 0$ . However, we do not prove this.

We now know that  $S_s$  is an arc only if  $s$  is oddly even. The rest of the chapter is devoted to proving that  $S_s$  is an arc when  $s$  is oddly even.

## 5.2 A Criterion for Arcs

Say that a *marked piece* is an compact, embedded, convex set with two distinguished vertices. Say that a *chain* is a finite union  $D_1, \dots, D_n$  of marked pieces such that  $D_i \cap D_{i+1}$  is one point, and that this point is one of the marked points on each of  $D_i$  and  $D_{i+1}$ . We also require that  $D_i \cap D_j = \emptyset$  for all other indices  $i \neq j$ . We define the *mesh* of the chain to be the maximum diameter of one of the marked pieces.

We say that a compact set  $S$  *fills* a chain  $D_1, \dots, D_n$  if  $S \subset \bigcup D_i$  and  $S$  contains every marked point of the chain. The purpose of this section is to establish the following (certainly well known) criterion.

**Lemma 5.4 (Arc Criterion)** *Let  $S$  be a compact set. Suppose, for every  $\epsilon > 0$ , that  $S$  fills a chain having mesh less than  $\epsilon$ . Then  $S$  is an embedded arc.*

We will assume that  $S$  satisfies the hypotheses of the lemma, and then show that  $S$  is an arc. First of all,  $S$  is clearly connected.

**Lemma 5.5** *Suppose that  $S$  fills a chain  $C_1, \dots, C_m$ . Then there is some  $\epsilon > 0$  with the following property. If  $S$  also fills a chain  $D_1, \dots, D_n$  having mesh size less than  $\epsilon$ , then  $S$  fills a chain  $E_1, \dots, E_p$  where each  $E_i$  has the form  $C_j \cap D_k$ .*

**Proof:** We can choose  $\epsilon$  so small that each  $D_j$  has following properties.

- The diameter of  $D_j$  is smaller than the length of any edge of any  $C_i$ .
- $D_j$  intersects any  $C_i$  in at most 2 edges.
- $D_j$  cannot intersect  $C_i$  and  $C_k$  if  $i$  and  $k$  are not consecutive indices.

For each  $i$ , there are unique and distinct pieces  $D_{j_1}$  and  $D_{j_2}$  which contain the two marked points of  $C_i$ . We claim that the pieces between  $D_{j_1}$  and  $D_{j_2}$  must have both marked points inside  $C_i$ . Assuming that this is true, we form the portion of the  $E$ -chain inside  $C_i$  by taking the intersections  $C_i \cap D_j$  and using the marked points of  $D_j$  for  $j_1 < j < j_2$ . The marked points of  $C_i \cap D_{j_1}$  are the marked point of  $C_i$  inside of  $D_{j_1}$  and the marked point of  $D_{j_1}$  inside  $C_i$ . Similarly for  $C_i \cap D_{j_2}$ . We do the same thing for each  $i$  and this gives us the conclusion of the Lemma.



Now we establish our claim. If our claim was false, then some  $D_j$ , with  $j_1 < j < j_2$ , would have one marked point in  $C_i$ . Note that  $D_j$  must have another marked point in either  $C_{i-1}$  or  $C_{i+1}$ , because this marked point is a vertex of  $S$ . Suppose that  $D_j$  has its other marked point in  $C_{i+1}$ . Then  $D_j \cap (C_i \cup C_{i+1})$  is disconnected because  $D_j$  does not contain the vertex  $C_i \cap C_{i+1}$ . Hence

$$D_1 \cup \dots \cup D_{j-1} \cup (D_j \cap (C_i \cup C_{i+1})) \cup D_{j+1} \cup \dots \cup D_n$$

consists of two disconnected components, each of which intersects  $S$  nontrivially. This contradicts the connectivity of  $S$ . ♠

If the chain  $C_1, \dots, C_m$  and the chain  $E_1, \dots, E_p$  are related as in the previous lemma, we say that  $E_1, \dots, E_p$  *refines*  $C_1, \dots, C_m$ . In view of the previous result, we can assume that  $S$  fills an infinite sequence  $\{\Omega_i\}$  of chains such that each one refines the previous one and the mesh size tends to 0.

For each  $i$ , we inductively create a partition  $P_i$  of  $[0, 1]$  into intervals, such that the number of intervals coincides with the number of marked pieces in  $\Omega_i$ , in the following manner. Once  $P_i$  is created, we distribute the intervals of  $P_{i+1}$  according to how  $\Omega_i$  contains  $\Omega_{i+1}$ . If the  $k$ th piece of  $\Omega_i$  contains  $n_k$  pieces of  $\Omega_{i+1}$ , then  $P_{i+1}$  is created from  $P_i$  by subdividing the  $k$ th interval of  $P_i$  into  $n_k$  intervals of equal size. Note that the mesh size of  $P_i$  tends to 0 as  $i$  tends to  $\infty$ .

There is a bijective correspondence between marked pieces in the chains and intervals in the partition. The correspondence respects the containment and intersection properties. For instance, two marked pieces intersect if and only if the corresponding intervals share an endpoint. Each point of  $S$  is contained in an infinite nested intersection of marked pieces, and we map this point to the corresponding nested intersection of intervals. This map is clearly a homeomorphism. The inverse map gives a parameterization of  $S$  as an arc in the plane.

### 5.3 Elementary Properties of the Limit Set

Let  $A_s$  be the set from §2.7.

**Lemma 5.6**  *$S_s$  contains the two left vertices of  $X_s$  and the two obtuse vertices of  $A_s$ .*

**Proof:** Let  $v$  be one of left vertices of  $X_s$ . Since the angle of  $X_s$  at  $v$  is not a right angle, there must be infinitely many squares contained in every neighborhood of  $v$ .

Note that the top left vertex of  $X_s^0$  is also the top obtuse vertex of  $A_s$ . So,  $S_s$  contains the top obtuse vertex of  $A_s$ . By symmetry,  $S_s$  contains the bottom obtuse vertex of  $A_s$ . ♠

By Lemma 2.7,  $\Delta_s$  consists entirely of squares. As we remarked after proving Lemma 2.7, these squares are either boxes or diamonds.

**Lemma 5.7** *Suppose  $\gamma \subset X_s^0$  is a compact arc which connects a point in a box to a point in a diamond. Then  $\gamma$  contains a point of  $S_s$ .*

**Proof:** Compare [S0, §8]. By compactness, it suffices to show that arbitrarily small perturbations of  $\gamma$  contain points of  $S_s$ . Hence, we may perturb so that  $\gamma$  does not contain any vertices of any tiles in  $\Delta_s$ . Suppose  $\gamma$  does not intersect  $\widehat{\Lambda}$ . Then  $\gamma$  only intersects finitely many tiles,  $\tau_1, \dots, \tau_n$ . Moreover,  $\tau_i$  and  $\tau_{i+1}$  must share an edge. Hence, by induction,  $\tau_1$  is a box if and only if  $\tau_n$  is a box. But  $\tau_1$  is a box and  $\tau_n$  is a diamond. This is a contradiction. ♠

**Lemma 5.8** *Each fundamental line of symmetry contains a point of  $S_s$ .*

**Proof:** To make the argument cleaner, we attach a large diamond  $\delta$  to the picture along the left edge of  $X_s$ , and we attach a large box  $\beta$  to the picture along the bottom edge of  $X_s$ . These extra squares are disjoint from  $X_s$  except along the relevant edges. Once we add these two squares, we see that each of the lines in question connects a diamond to a box.  $H$  connects  $\delta$  to the leftmost central tile of  $\Delta_s$  and both  $V, D, E$  all connect  $\beta$  to  $\delta$ . By Lemma 5.7, each of these lines contains a point of  $S_s$ . ♠

## 5.4 The End of the Proof

Suppose that  $S_s$  fills some chain  $D_1, \dots, D_n$ . We call this chain *good*

- $D_j$  is disjoint from the interiors of the edges of  $\partial A_s$ , for  $j = 1, \dots, n$ .
- The first marked point of  $D_1$  is the bottom left vertex of  $X_s$ .

- The last marked point of  $D_n$  is the top left vertex of  $X_s$ .

**Lemma 5.9**  $S_s$  fills a good chain.

**Proof:** Our chain has two pieces. We set  $D_1 = B_s$ , the triangle from §2.7, and we let  $D_2$  be the set from Corollary 4.3. ♠

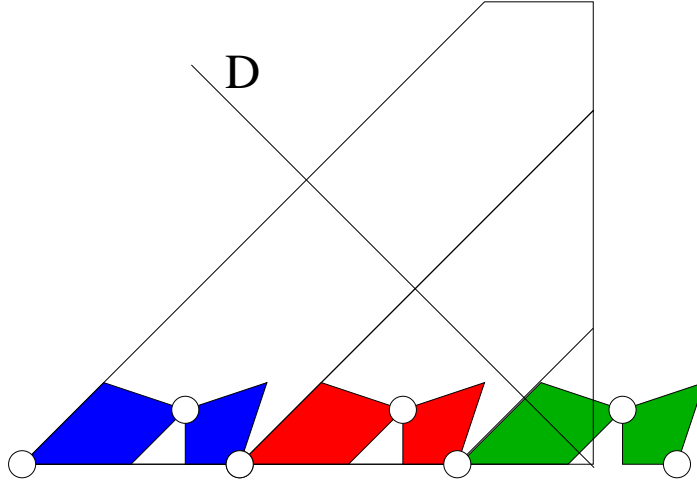
**Lemma 5.10** Let  $t = R(s)$ . Suppose  $S_t$  fills a good chain having mesh  $m$ . Then  $S_s$  fills a good chain having mesh at most  $m/\sqrt{2}$ .

**Proof:** Let  $\Omega_t$  be the good chain filled by  $S_t$ . We use the notation from the Filling Lemma, Equation 8, and Theorem 2.5. We make our construction in 5 steps.

**Step 1:** For  $j = 0, \dots, K$ , we define

$$\Omega_j = T_s^j \circ \phi_s(\Omega_t). \quad (22)$$

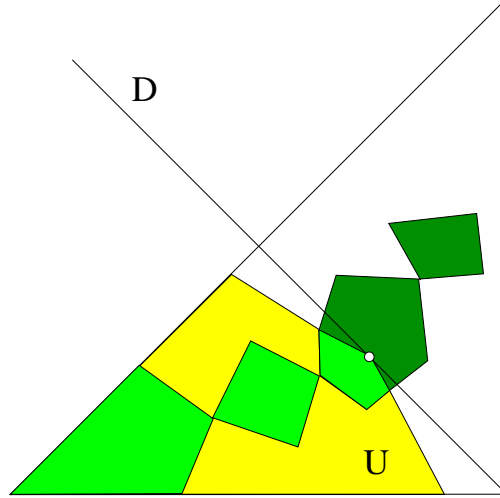
Figure 5.1 illustrates our construction. The individual chains  $\Omega_0, \dots, \Omega_{K-1}$  piece together to make one long chain because the second disk of  $\Omega_j$  touches the common edge between  $\Psi_s^j$  and  $\Psi_s^{j+1}$  only at the bottom vertex of this edge.



**Figure 5.1:** Step 1: The chains  $\Omega_j$  for  $j = 0, \dots, K$ . Here  $K = 2$ .

**Step 2:** The problem with  $\Omega_K$  is that some of it sticks over the edge of  $X_s^0$ . This is the green set in Figure 5.1. However, we know from the Pinching Lemma that  $S_s$  intersects the line  $D_s$  in a single point. All other points of  $D_s$  must have neighborhoods contained in finitely many squares. For this reason, we can make the essentially the same construction as in Corollary 4.3 to produce a convex disk  $U \subset \Psi_s^K$  such that  $S_s \cap \Psi_s^K \subset U$  and  $U \cap D_s$  is the single point which belongs to  $S_s$ .

We now improve  $\Omega_K$  as follows. We intersect each piece of  $\Omega_K$  with the set  $U$  and throw out all those after the first one which is disjoint from  $U$ .



**Figure 5.2:** Step 2: Improving  $\Omega_K$ .

The result is a chain which joins the bottom vertex of the edge  $\Psi_s^{K-1} \cap \Psi_s^k$  to the point  $S_s \cap D_s$ . Figure 5.2 shows the construction. We call this improved chain  $\Omega'_K$ .

Define

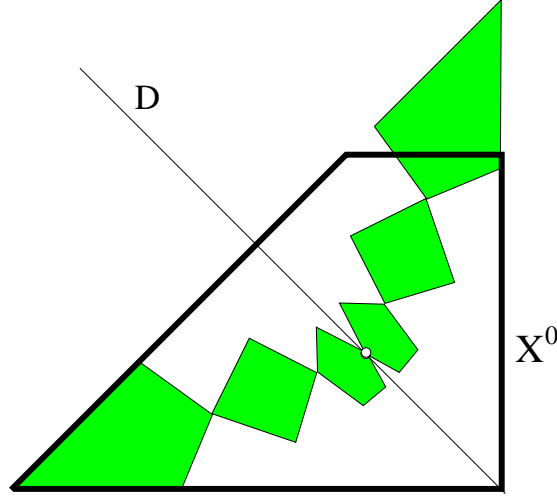
$$\Upsilon_0 = \Omega_0, \dots, \Omega_{K-1}, \Omega'_K. \quad (23)$$

By construction, this chain is filled by the portion of  $S_s$  beneath the line  $D_s$ .

**Step 3:** Define

$$\Upsilon_2 = \Upsilon_0, \Upsilon_1; \quad \Upsilon_1 = R_D(\Upsilon_0) \quad (24)$$

That is, we continue our chain by reflecting it across the line  $D$ . The resulting chain contains  $S_s$ , by symmetry, but we are not quite done.



**Figure 5.3:** Step 3: Extending by reflection

Some of the final pieces of  $\Upsilon_1$  might not lie in  $X_s$ . The problem is that  $X_s$  is not symmetric with respect to  $R_D$ . The portion below  $D$  is larger than the portion above  $D$ .

**Step 4:** We finish the construction by a method very similar to what we did in Step 2. We simply intersect all the pieces of  $\Upsilon_1$  with  $X_s^0$ , and let  $\Upsilon'_1$  and omit all those pieces which come after the first one which has trivial intersection with  $X_s^0$ . We set  $\Upsilon_3 = \Upsilon_0, \Upsilon'_1$ . By construction,  $\Upsilon_3$  is a chain filled by  $S_s$ . Moreover, since  $\phi_s$  contracts distances by some  $\lambda_s < 1/\sqrt{2}$ , we see that the mesh of  $\Upsilon_3$  is less than  $m/\sqrt{2}$ .

**Step 5:** The chain  $\Upsilon_3$  might not be clean. To remedy this, we shrink the pieces slightly (away from the marked points) so that they are all disjoint from the interiors of the edges of  $A_s$ . What allows us to do this is the Shield Lemma combined with compactness. The final chain has all the desired properties. ♠

Note that if  $s$  is oddly even, then so is  $R(s)$ . The chains in Lemma 5.9 all have mesh size less than 2. It now follows from iterating Lemma 5.10 that  $S_s$  fills a good chain having mesh size less than  $\epsilon$ , for any given  $\epsilon_0$ . Our Arc Criterion now shows that  $S_s$  is an arc. This completes the proof of Statement 1 of the Main Theorem.

## 6 Proof of Statement 2

### 6.1 The Main Argument

In this chapter we suppose throughout that  $s \in (0, 1)$  is irrational and  $R^n(s) > 1/2$  only finitely often. This means that there is some  $n$  such that  $R^n(s)$  is oddly even. Let  $f(s)$  denote the smallest  $n$  with this property. Let  $S_s$  denote the left half of  $\widehat{\Lambda}_s$ , as in the previous chapter. Our goal is to show that  $S_s$  is a finite forest.

**Lemma 6.1** *Both  $S_s \cap A_s$  and  $S_s \cap B_s$  are finite unions of arcs.*

**Proof:** When  $f(s) = 0$ , the result is true by Statement 1 of the Main Theorem. Suppose by induction that this lemma is true for all  $s$  such that  $f(s) < N$ . If  $s$  is chosen so that  $f(s) = N$ , then let  $t = R(s)$ . Then  $f(t) = N - 1$ . By induction, both  $S_t \cap A_t$  and  $S_t \cap B_t$  are finite unions of arcs. By the Filling Lemma, both  $\widehat{\Lambda}_s \cap A_s$  and  $\widehat{\Lambda}_s \cap B_s$  are contained in finite unions of similar copies of  $\widehat{\Lambda}_t \cap A_t$  and  $\widehat{\Lambda}_t \cap B_t$ . The argument is very similar to what we did to show that  $S_s$  is connected in the oddly even case. Hence, both  $S_s \cap A_s$  and  $S_s \cap B_s$  are finite unions of arcs.

**Corollary 6.2**  *$S_s$  is a finite union of arcs.*

**Proof:** We have  $S_s = (S_s \cap A_s) \cup (S_s \cap B_s)$ . ♠

Following this section, the entire chapter is devoted to proving the following result.

**Theorem 6.3 (No Loops)** *Let  $s \in (0, 1)$  be any irrational number. Then  $S_s$  contains no embedded loops.*

But a finite union of arcs which contains no embedded loops must be a finite forest. Modulo the No Loops Theorem, this completes of Statement 2 of the Main Theorem.

We prove the No Loops Theorem using the same kind of extremality argument we gave for the Pinching Lemma. One case of this argument is much harder than the others, and so we split it off and tackle it first. Once we build the machinery for this one case, the rest of the proof is easy.

## 6.2 Interaction with the Octagrid

Suppose that  $s < 1/2$  and  $t = R(s) < 1/2$ . We consider the octagrid components defined in §3.3. Here is the main result of this section.

**Lemma 6.4** *An embedded loop in  $S_s$  lies in a single octagrid component.*

We will prove this through a series of smaller results. Say that an *octagrid edge* is an edge of one of the octagrid components. Let  $u = R(t)$ .

**Lemma 6.5**  *$S_s$  intersects each octagrid segment in at most one point provided that  $u < 1/2$ .*

**Proof:** Let  $G$  be an octagrid component and let  $\sigma$  be an edge of  $G$ . Just as in the proof of Lemma 3.3, we can use bilateral symmetry and the map  $T_s$  from Equation 8 to reduce to the case where  $G \subset Z_s^0$ . If one endpoint of  $\sigma$  is contained in a square of the pyramid associated to  $\Delta_s$ , then  $G$  is entirely contained in that square. This case is trivial. So,  $\sigma$  must be one of the 8 segments emanating from the center of the bottom left tile of the extended pyramid associated to  $\Delta_s$ .

By Theorem 2.5, it suffices to prove that  $\sigma' = \phi_s^{-1}(\sigma)$  intersects at most one point of  $S_t$ , where  $t = R(s)$ . One endpoint of  $\sigma'$  is the center  $c'$  of the central tile of  $Z_t$ . We think of  $\sigma'$  as pointing away from  $c'$ . Since  $u < 1/2$ , the central tile of  $Z_t$  is a square. The 4 nontrivial cases are as follows.

1.  $\sigma'$  is horizontal and points left.
2.  $\sigma'$  has slope 1 and points down.
3.  $\sigma'$  is vertical and points down.
4.  $\sigma'$  has slope 1 and points up.

In the other 4 cases,  $\sigma'$  is entirely contained in the central tile of  $Z_t$ . In the 4 nontrivial cases,  $\sigma'$  is longer than an octagrid edge associated to the parameter  $t$ , but this does not bother us.

The line  $D_t$  of symmetry contains the point  $c'$ . Reflection in  $D_t$  reduces Case 3 to Case 1 and Case 4 to Case 2. So, it suffices to consider Cases 1 and 2. In Case 1,  $\sigma'$  is a segment of  $H^0$ . In this case, we apply the Pinching Lemma. In Case 2, we apply Theorem 2.5 again:  $\sigma'' = \phi_t^{-1}(\sigma')$  is the left half of  $H$  and intersects  $S_u$  at most once by the Pinching Lemma. ♠

**Lemma 6.6**  $S_s$  intersects each octagrid segment in at most one point provided that  $u > 1/2$ .

**Proof:** The proof here is the same, except that the central tile of  $Z_t$  is an octagon. In this case, we immediately renormalize so that  $\sigma'' = \phi_t^{-1}(\sigma')$  is one of the 8 segments emanating from the center  $(0,0)$  of the trivial tile of  $\Delta_u$ . After reflecting in the origin, we arrive at 2 nontrivial cases.

1.  $\sigma''$  is horizontal and points left.
2.  $\sigma''$  has slope 1 and points downward.

In the first case  $\sigma''$  is simply the portion of  $H$  lying to the left of the origin. This case follows from the Pinching Lemma applied to the parameter  $u$ . In the second case,  $\sigma''$  agrees with the line  $E_u$  of symmetry outside the trivial tile of  $\Delta_u$ . Again, this follows from the Pinching Lemma applied to the parameter  $u$ . ♠

We record the obvious corollary.

**Corollary 6.7**  $S_s$  intersects each octagrid segment in at most one point.

Suppose now that  $\gamma$  is an embedded loop in  $S_s$ .

**Lemma 6.8**  $\gamma$  cannot link any vertex of an octagrid component.

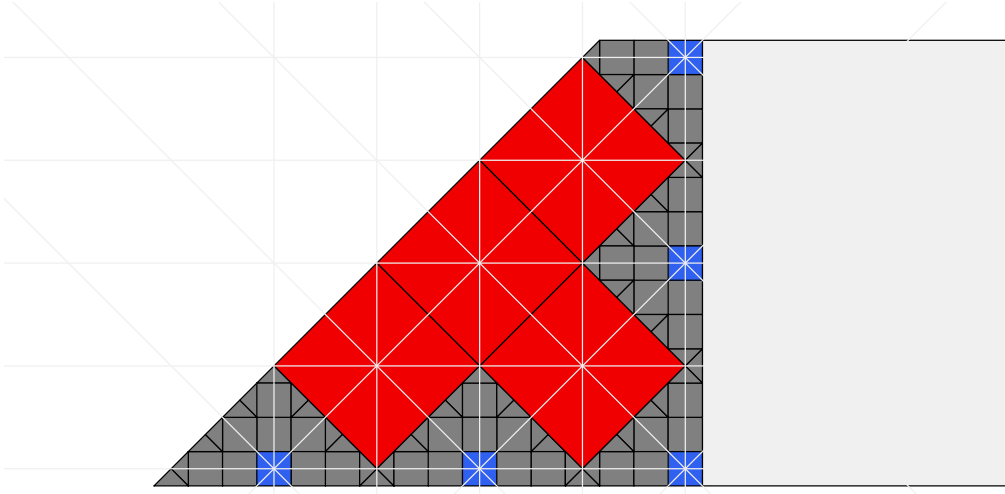
**Proof:**  $\gamma$  cannot surround any square in the pyramid associated to  $s$ , because this pyramid is a topological disk sharing an arc with  $\partial X_s$ .

Say that a *peripheral tile* is a tile of the extended pyramid which is not a tile of the pyramid. The peripheral tiles are colored blue in Figures 5.8 and 5.9.

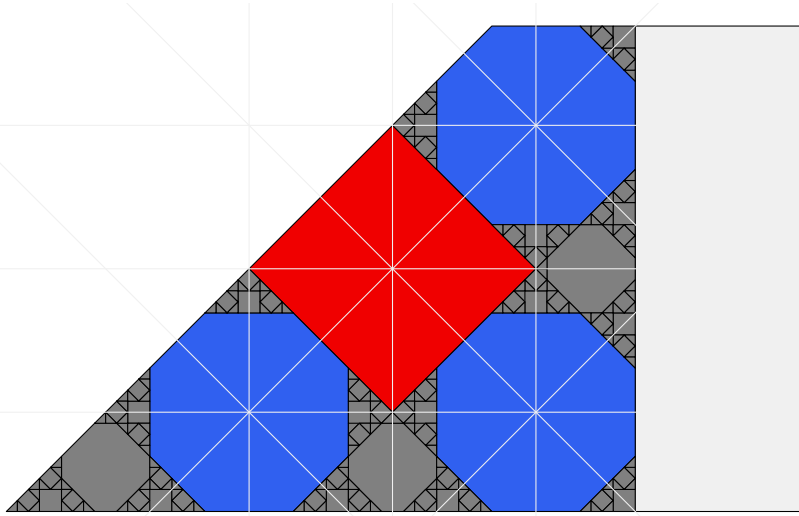
Say that a *blocker* is an octagrid edge which lies entirely in a tile and has a vertex in  $\partial X_s^0$ . These segments are disjoint from  $S_s$  and hence from  $\gamma$ . Each peripheral tile has at least one blocker. But then  $\gamma$  cannot surround a peripheral tile because it would have to intersect a blocker.

The remaining vertices of the octagrid (meaning, the vertices of the octagrid components) have the following structure. At least one octagrid edge emanating from the vertex lies entirely inside a tile. For this reason,  $\gamma$  cannot link any vertex of an octagrid component. ♠





**Figure 6.1:** The octagrid for  $s = 13/32$ .



**Figure 6.2:** The octagrid for  $s = 22/57$ .

If  $\gamma$  is not contained in a single octagrid component, and  $\gamma$  does not link any of the vertices of octagrid components, then  $\gamma$  must cross some octagrid edge twice. We have already ruled this out. Hence  $\gamma$  is contained in a single octagrid component. This completes the proof of Lemma 6.5.

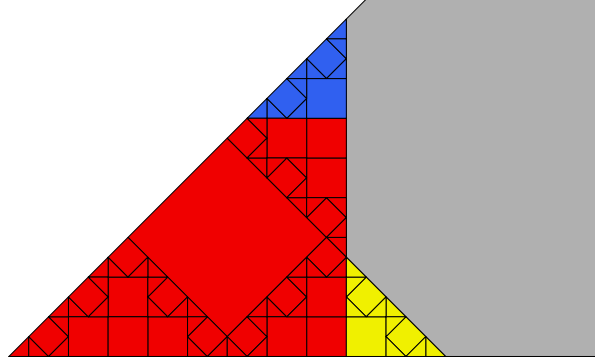
### 6.3 No Embedded Loops

Now we prove the No Loops Theorem. Our method of proof is very much like what we did for the Pinching Lemma. Say that a *counterexample* is a pair  $\Omega = (\gamma, s)$  where  $\gamma$  is an embedded loop in  $S_s$ . We define

$$\lambda(\Omega) = \frac{\text{diam}(\gamma)}{g(s)}, \quad g(s) = \text{diam}(X_s^0). \quad (25)$$

**Lemma 6.9** *For any  $\epsilon > 0$  there exists a counterexample  $\Omega$ , whose parameter is less than  $1/2$ , such that  $\lambda(\Omega) > \lambda - \epsilon$ .*

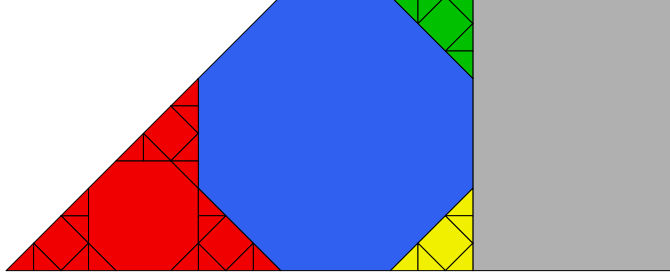
**Proof:** As in the proof of Lemma 4.5, it suffices to consider the case of a counterexample  $\Omega$  having  $s \in (1/2, \sqrt{2}/2]$ . In this case, the reflection  $R_V$  maps the region to the right of  $V$  into  $Z_s^0$  and the reflection  $R_D$  maps the region above  $D_s$  into  $Z_s^0$ . Figure 6.3 shows a fairly typical example.



**Figure 6.3:**  $Z_s^0$  (red) and the components of  $X_s^0 - Z_s$  for  $s = 11/17$ .

By the Pinching Lemma,  $\gamma$  can intersect each of  $V$  and  $D_s$  at most once. Hence  $\gamma$  lies to one side or the other of each of these lines. So, by symmetry, we can assume that  $\gamma \subset Z_s^0$ . The rest of the proof is as in Lemma 4.5. ♠

As in the proof of the Pinching Lemma, we now analyze potential counterexamples having parameter  $s \in (1/4, 1/2)$ . First suppose  $s \in (1/4, 1/3)$ . For  $s$  in this range, we have  $t = R(s) > 1/2$ . Let  $O_s$  be the image, under the  $\phi_s^0$  of the central tile of  $\Delta_s$ . This octagon separates  $X_s^0$  into three regions having disjoint closures, as shown in Figure 6.3. One of the regions, colored red in Figure 6.3, is precisely  $Z_s^0$ .



**Figure 6.4:**  $Z_s^0$  (red) and other regions for  $s = 5/17$ .

The reflections  $R_D$  and  $R_V$  respectively carry the other regions into the one contained in  $Z_s$ . Hence, we may assume by symmetry that  $\gamma \subset Z_s$ . We now get the same contradiction as we got in the proof of the Pinching Lemma.

For  $s \in (1/3, 1/2)$  we have  $R(s) < 1/2$ . Lemma 6.5 applies, and so we know that our counterexample is contained in a single octagrid component. But then, by Lemma 3.3, we can assume by symmetry that our counterexample lies in  $Z_s$ . This gives us the same contradiction as in the Pinching Lemma.

This completes the proof that  $\widehat{\Lambda}_s$  has no embedded loops.

## 7 Proof of Statement 3

### 7.1 The Limit Set is Perfect

Recall that a closed set  $C$  is *perfect* if every point  $p \in C$  is an accumulation point of  $C - \{p\}$ .

**Lemma 7.1** *When  $s$  is irrational,  $\widehat{\Lambda}_s$  is a perfect set.*

**Proof:** If  $\widehat{\Lambda}_s$  is not perfect, then there is some  $p \in \widehat{\Lambda}_s$  and some open disk  $U$  containing  $p$  such that  $U \cap \widehat{\Lambda}_s = \{p\}$ . The open set  $U$  must contain infinitely many tiles of  $\Delta_s$ , because  $\widehat{\Lambda}_s \cap U$  is nonempty. Therefore, if we write  $\Delta_s$  as in the Covering Lemma, the image of some patch must have  $p$  as an accumulation point. Choosing  $\epsilon$  small enough, we can guarantee that there exists an  $\epsilon$ -patch  $(K, \psi, \epsilon)$  such that  $\psi(K) \subset U$ .

If  $K$  is a triangle, then two of the vertices  $v_1$  and  $v_2$  of  $K$  have acute angles. (The angle is  $\pi/4$ .) These vertices must be accumulation points of infinitely many tiles, because all the tiles are squares and semi-regular octagons. But then  $\psi(v_1)$  and  $\psi(v_2)$  are accumulation points of infinitely many squares in  $\Delta_s$ . Hence  $\widehat{\Lambda}_s \cap U$  contains at least 2 points. This is a contradiction.

If  $K$  is a pentagon, then  $K$  has 2 vertices  $v_1$  and  $v_2$  with obtuse angles. (The angle is  $3\pi/4$ .) If  $v_1$  is not an accumulation point of infinitely many tiles of  $\Delta_u \cap K$ , then  $v_1$  is the vertex of some octagon of  $\Delta_u \cap K$ . But then there are two new acute vertices  $w_1$  and  $w_2$  which must be accumulation points of infinitely many tiles of  $\Delta_u \cap K$ . This gives us the same contradiction as above. The only way out of the contradiction is for both  $v_1$  and  $v_2$  to be accumulation points of infinitely many tiles of  $\Delta_u \cap K$ , but this is again a contradiction. There is no way out. ♠

### 7.2 Overview for the Rest of the Proof

We call  $s \in (0, 1)$  an *octagonal parameter* if  $R^n(s) > 1/2$  infinitely often. The reason for the name is that, thanks to Theorem 1.1,  $\Delta_s$  contains infinitely many octagons if and only if  $s$  is octagonal. Our remaining goal is to prove that  $\widehat{\Lambda}_s$  is a Cantor set when  $s$  is octagonal. Let  $S_s$  denote the left half of  $\widehat{\Lambda}_s$ , as usual.

We already know that  $S_s$  is closed and perfect. It remains to show that  $S_s$  is totally disconnected for any octagonal  $s$ . The proof is a bootstrap argument. The basic idea is that renormalization tends to make connected components of  $S_s$  larger. So, we will start with the assumption that  $S_s$  has a nontrivial connected component when  $s$  is octagonal, and ultimately we will produce a new octagonal parameter  $u$  for which  $S_u$  has a very large and egregious kind of connected component, which we can rule out. Throughout the proof,  $K_s$  will stand for a *symmetric piece*, one of the sets  $A_s, B_s, P_s, Q_s$  from §2.7.

### 7.3 Unlikely Sets

**Lemma 7.2** *Let  $s \in (0, 1)$  be irrational. Let  $K_s$  be a symmetric piece. Let  $e$  be any edge of  $K_s$ . Then some tile of  $\Delta_s \cap K_s$  has an edge in  $e$ .*

**Proof:** We do this in a case-by-case way.

1. Suppose  $s < 1/2$  and  $K_s \in \{B_s, Q_s\}$ . Applying Theorem 2.5 repeatedly, we see that there are infinitely many tiles which have edges in the bottom edge of  $X_s$  and infinitely many tiles which have edges in the left edge of  $X_s$ . Eventually these tiles lie in both  $B_s$  and  $Q_s$ . This takes care of two out of three edges of  $B_s$  and  $Q_s$ . The third edge, in each case, follows from bilateral symmetry.
2. Suppose  $s < 1/2$  and  $R(s) > 1/2$  and  $K_s \in \{A_s, P_s\}$ . In this case, there is an octagon having the same width as the central tiles, and this octagon has edges in all the sides of  $A_s$ . Likewise, the same octagon has edges in all the sides of  $P_s$ . See Figure 4.12.
3. Suppose  $s < 1/2$  and  $R(s) < 1/2$  and  $K_s \in \{A_s, P_s\}$ . In this case, each edge of  $A_s$  has a segment which is an edge of one of the tiles in the extended pyramid. The same goes for  $P_s$ . See Figures 3.8 and 3.9.
4. Suppose  $s > 1/2$ . The argument given in Case 1 takes care of  $P_s$  and  $B_s$ . The case of  $A_s$  and  $Q_s$  follows from inversion symmetry. See Remark (i) in §2.7.

Thus exhausts the possibilities. ♠

Let  $C$  be a nontrivial connected subset of  $\widehat{\Lambda}_s$ . Let  $\mathcal{K}$  be a patch cover, as in the Covering Lemma. What we mean is that  $\mathcal{K}$  is a finite union of patches and tiles, as in Equation 20. We call  $C$  *bad* with respect to  $\mathcal{K}$  if  $C$  does not intersect the interiors of the images of any of the patches. That is,  $C$  is disjoint from the interiors of all the sets  $\psi_j(K_j)$ .

Call  $C$  *unlikely* if, for every  $\epsilon > 0$ , there is a patch covering  $\mathcal{K}$  of scale less than  $\epsilon$ , so that  $C$  is bad with respect to  $\mathcal{K}$ . So,  $C$  is bad with respect to an infinite sequence of patch covers, having scale tending to 0.

**Lemma 7.3** *Let  $s \in (0, 1)$  be irrational. There do not exist any unlikely subsets of  $\widehat{\Lambda}_s$ .*

**Proof:** We will suppose some unlikely set  $C$  exists and get a contradiction. By taking a suitable subset of  $C$ , we can assume that  $C$  is a line segment contained in the boundary of some particular patch of one of the sequence of bad covers. By Lemma 7.2, some segment of  $C$  lies in a tile boundary. Further shrinking  $C$ , we can assume that  $C$  is one edge of some tile  $\tau_1$ .

The midpoint  $m$  of  $C$  is the accumulation point of infinitely many tiles of  $\Delta_s$ . These tiles are all disjoint from  $\tau_s$ . Hence, there is some patch  $(\psi, K, u)$  so that  $m \in \psi(K)$ . But  $C$  cannot intersect the interior of  $\psi(K)$ . Hence, one edge of  $\psi(K)$  lies in the line containing  $C$ . Shrinking the scale as needed, we can assume that one edge of  $\psi(K)$  is contained in  $C$ .

Some tile  $\tau_2$  of  $\psi(K) \cap \Delta_s$  has an edge in  $C$ , by Lemma 7.2. But then some point of  $C$  lie on the common boundary of  $\tau_1$  and  $\tau_2$  and cannot belong to the limit set. This is a contradiction. ♠

## 7.4 Tails

For each symmetric set  $K_u \in \{A_u, B_u, P_u, Q_u\}$ , and each subset  $C \subset S_u$ , let  $C \sqcap K_u$  denote the set of points  $p \in C$  such that every neighborhood of  $p$  contains infinitely many tiles of  $K_u \cap \Delta_u$ . Note that  $C \sqcap K_u$  might be a proper subset of  $C \cap K_u$ , but the two sets agree on the interior of  $K_u$ . The former set is easier to pull back using the patches from the Covering Lemma.

We say that the symmetric piece  $K_u$  has a *tail* if some connected component of  $S_u \sqcap K_u$  contains both an interior point of  $K_u$  and a boundary point of  $K_u$ . Conceptually, we think of a tail as a little arc which joins a

boundary point of  $K_u$  to an interior point, but we don't actually know that our connected set is path connected.

**Lemma 7.4** *Suppose  $S_s$  is not totally disconnected. Then for all sufficiently large  $n$  the parameter  $u = R^n(s)$  has the following property. At least one of the 4 symmetric pieces  $K_u$  has a tail.*

**Proof:** If  $S_s$  is not totally disconnected, then it has a closed connected subset  $C$  having more than one point. By Lemma 7.3, once  $n$  is large enough and  $u = R^n(s)$ , we can find a patch  $(K, \psi, u)$  such that some point  $p \in C$  lies in the interior of  $K' = \psi(K)$  and some point of  $C$  lies outside of  $K'$ .

Consider the set  $C \cap K'$ . For the sake of exposition, assume first that  $C$  is path connected. Then we can find some path  $\alpha \in C$  joining  $p$  to some point of  $C$  lying outside  $K'$ . Let  $\beta$  be the maximal initial portion of  $\alpha$  which lies in  $K'$ . By construction  $\beta \subset C \cap K'$  and  $\beta$  joins  $p$  to a point  $q \in \partial K'$ . By definition of a patch,  $\psi^{-1}(\beta)$  is a path in  $S_u \cap K_u$  joining  $\psi^{-1}(p)$  to  $\psi^{-1}(q)$ . The former point lies in the interior of  $K$  and the latter point lies on the boundary. This gives  $K_u$  a tail.

We follow the same outline when  $C$  is merely connected. Define an  $\epsilon$ -chain to be a sequence of points  $p_0, \dots, p_N \in C$  such that  $\|p_k - p_{k+1}\| < \epsilon$  for all  $k$ . We say that this chain joins  $p_0$  to  $p_N$ . Any open neighborhood of a connected set is path connected. Hence, for any  $m$ , there is a  $(1/m)$ -chain  $\alpha_m$  joining  $p$  to some point of  $C$  outside of  $K'$ . Let  $\beta_m$  be the maximal initial portion of  $\alpha_m$  which lies entirely in  $K'$ . Passing to a subsequence, we can assume that the endpoints of  $\beta_n$  converge. One of the endpoints is always  $p$ . Let  $q$  be the limit of the other endpoints. By construction  $q \in \partial K'$ .

Let  $\beta$  be the connected component of  $C \cap K'$  containing  $p$ . By construction  $q \in C \cap K'$ . Moreover, for every  $\epsilon > 0$  there is an  $\epsilon$ -chain connecting  $p$  to  $q$ . Hence  $q \in \beta$ . Now we pull back by  $\psi$  to give  $K_u$  a tail, as in the path connected case. ♠

**The Chain Trick:** We will have many occasions below to make the same kind of argument as we just gave, and the geometry behind the argument is always clearer in the path connected case. For this reason, we will give the arguments in the path connected case and then remark that the same trick as used above – i.e., using an infinite sequence of chains in place of a path – handles the general case. For reference, we call this the *Chain trick*:

## 7.5 Crosscuts

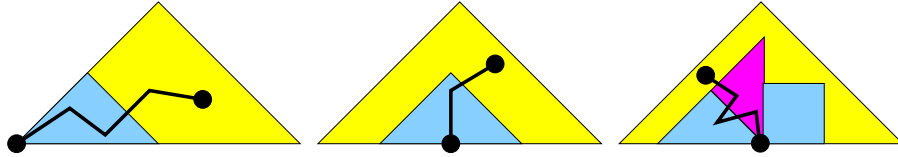
Let  $K$  be a (solid) triangle and let  $\Lambda \subset K$  be a closed set. We say that a *crosscut* for  $(K, \Lambda)$  is a connected subset of  $\Lambda$  which contains two points  $p, q \in \partial K$ . We require that  $p$  and  $q$  are not both contained in the interior of the same edge of  $K$ .

In the following result, it is not really essential that  $s$  is octagonal and that  $s, u > 1/2$ . However, in this case, the symmetric pieces associated to the two parameters are all right-angled isosceles triangles. This makes the geometry easier and cuts down on the number of cases to consider.

**Lemma 7.5** *Suppose that  $s > 1/2$  is octagonal and  $S_s$  is not totally disconnected. Then for all sufficiently large  $n$  the parameter  $u = R^n(s)$  has the following property provided that  $u > 1/2$ : The pair  $(K_u, K_u \sqcap S_u)$  has a crosscut for some symmetric piece  $K_u$ .*

**Proof:** By Lemma 7.4, we can assume without loss of generality that some  $K_s$  has a tail  $C$ . As remarked above, we will assume that  $C$  is a path; the Chain Trick then finishes the proof.

Let  $\epsilon = \|p - q\|$  and choose  $n$  so large that the patch covering in the Covering Lemma corresponding to  $u = R^n(s)$  has scale much smaller than  $\epsilon$ . There is a patch  $(K_j, \psi_j, u)$  such that  $K' = \psi_j(K_j)$  contains  $p$ . There are several cases to consider, as shown in Figure 7.1. (The third part of Figure 7.1 just shows one of the several possibilities for that case.)



**Figure 7.1:** Possibilities for the patch covering.

- Suppose that  $p$  is a vertex of  $K_s$ . Then  $p$  is also a vertex of  $K' = \psi_j(K_j)$ . The two sides of  $K'$  incident to  $p$  lie in sides of  $K$ . But then the path  $C$  must exit through the third side of  $K'$  in order to reach  $q$ . The pullback  $\psi_j^{-1}(C)$  is a crosscut of  $K_j$ .
- Suppose  $p$  lies in the interior of an edge of  $K$  and also in the interior of an edge of  $K'$ . Then, again,  $C$  must exit  $K'$  through one of the other edges of  $K'$ . Now we pull back as before.



- Suppose  $p$  lies in the interior of an edge of  $K$  and is a vertex of  $K'$ . In this case, we can assume that  $p$  is not contained in the interior of an edge of the image any other patch. So, a neighborhood of  $p$  in  $K$  is covered by finitely many tiles of  $\Delta_s$  and either 1, 2, or 4 patch images, each of which has  $p$  as a vertex. In any case,  $\gamma$  must exit this neighborhood, and some initial portion of  $\gamma$  makes a crosscut in one of the patch images. Now we pull back as before.

This exhausts the possibilities. ♠

The rest of the proof involves ruling out the existence of crosscuts associated to octagonal parameters.

## 7.6 The Proof Modulo one Case

**Lemma 7.6** *Suppose that  $s > 1/2$  is an octagonal parameter and  $K_s$  is a symmetric piece with a crosscut. Then the crosscut cannot contain an acute vertex of  $K_s$ .*

**Proof:** Suppose the crosscut contains a vertex  $p$ . Using the various kinds of bilateral symmetry discussed in §2.7 we reduce to the case when  $p$  is the bottom left vertex of  $X_s$  and  $K_s$  is one of  $B_s$  or  $P_s$ . (These are the two pieces having this point as a vertex.)

By Corollary 5.2, there are infinitely many octagons wedged into  $X_s$ . These octagons shrink down to  $p$ , and isolate  $p$  from the rest of  $S_s$ . So,  $p$  is its own connected component of  $S_s$ . ♠

We call a crosscut of  $K_s$  *acute* if it contains points  $p, q$ , where  $p$  lies in the interior of a short side of  $K_s$  and  $q$  lies in the interior of a long side of  $K_s$ . We use this name, because we think of a path joining  $p$  to  $q$  and subtending one of the acute angles.

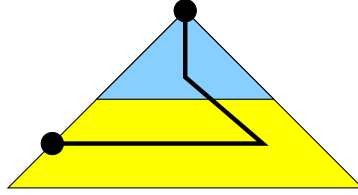
**Lemma 7.7 (Acute)** *Suppose that  $s > 1/2$  is an octagonal parameter and  $K_s$  is a symmetric piece with a crosscut. Then the crosscut cannot be acute.*

**Proof:** We prove this in the next section. ♠

Say that a *right crosscut* is a crosscut of  $K_s$  which contains the right-angled vertex of  $K_s$ .

**Lemma 7.8** *Suppose that  $s > 1/2$  is an octagonal parameter and  $K_s$  is a symmetric piece with a crosscut. The crosscut cannot contain the right-angled vertex of  $K_s$ .*

**Proof:** Let  $C$  be a crosscut which supposedly contains the right-angled vertex  $p$  of  $K_s$ . We will give the proof when  $C$  is a path, and then the Chain Trick handles the general case. Since  $s$  is octagonal, there are arbitrarily large choices of  $n$  for which  $u = R^n(s) > 1/2$ . If we choose  $n$  sufficiently large, then by the Covering Lemma we can find a patch  $(\psi_j, K_j, u)$  so that  $K' = \psi_j(K_j)$  has  $p$  as a vertex, shares two edges with  $K$ , and has diameter (say)  $\epsilon/10$ . This is shown in Figure 7.2.



**Figure 7.2:** The pieces  $K$  and  $K'$ .

But then  $C$  must exit  $K'$  from the hypotenuse. The maximal initial portion of  $C$  contained in  $K'$  is an acute crosscut (with respect to either acute angle). But, Lemma 7.7 says that these cannot exist. ♠

Before we give the final argument, we single out some important points.

**The Proper Nesting Property:** Suppose  $K' = \psi_j(K_j) \subset K_s$  is the image of some patch which arises in the conclusion of the Covering Lemma for the parameters  $s$  and  $u$ . If  $K'$  and  $K_s$  share a vertex, then this vertex has the same type (acute or right) with respect to both triangles. One sees this just by inspecting the equations used in the proof of Lemma 4.6. Call this the *proper nesting property*.

**Images of Crosscuts:** If  $(K_j, \psi_j, u)$  is a patch for the parameter  $s$ , and  $K_j$  has a right crosscut, then some connected subset  $S_s \cap K'$  contains points on the interiors of both the short edges of  $K' = \psi_j(K_j)$ . We will abuse our terminology and say that  $K'$  has a crosscut, even though technically  $K'$  is a similar copy of a symmetric piece associated to the different parameter  $u$ .

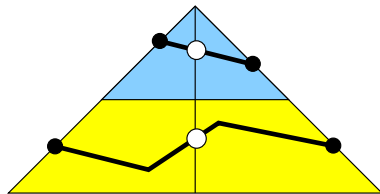
With this terminology, a crosscut for  $K'$  is a subset of  $S_s$ .

We call a crosscut of  $K_s$  *right* if it contains points  $p, q$ , where  $p$  and  $q$  respectively lie in the interiors of the two short sides of  $K_s$ . Modulo proving Lemma 7.7, we have ruled out all types of crosscut except right crosscuts. So, if Statement 3 of the Main Theorem is false, then there is an infinite sequence of octagonal parameters  $s_1, s_2, \dots$  with the following properties.

- $s_{k+1} = R^{n_k}(s_k)$  for some  $n_k > 0$ .
- Some symmetric piece  $K_{s_k}$  has a right crosscut for  $k = 1, 2, 3, \dots$
- The crosscut associated to  $K_{s_k}$  is disjoint from a neighborhood of the right-angled vertex of  $K_{s_k}$ . (Otherwise, we reduce to the previous case.)

Once one symmetric piece  $K_s$  has a crosscut, all similar copies of  $K_s$  have crosscuts. Moreover, by Lemma 4.6, each similar copy of a symmetric piece for the parameter  $s_j$  contains a similar copy of a symmetric piece for the parameter  $s_k$  as long as  $k > j$ .

Using the proper nesting property together with the Pidgeonhole Principle, we can find two nested (similar copies of) symmetric pieces which both have right crosscuts. We can arrange that the smaller piece is so small that its crosscut is completely disjoint from the crosscut of the larger piece, as shown in Figure 7.3.



**Figure 7.3:** Nested symmetric pieces

Call  $s$  the parameter associated to the larger of the two symmetric pieces. By construction, both the crosscuts we have found must cross the line of symmetry of  $K_s$ . Moreover, both crosscuts are subsets of  $S_s$ . Hence,  $S_s$  crosses the line of symmetry for  $K_s$  at least twice. This contradicts the Pinching Lemma.

The only way out of the contradiction is that Statement 3 of the Main Theorem is true. This completes the proof of the Main Theorem, modulo the proof of Lemma 7.7.

## 7.7 No Acute Crosscuts

Here we complete the proof of the Main Theorem by establishing Lemma 7.7. If some symmetric piece  $K_s$  has an acute crosscut, then we can use bilateral symmetry to reduce to the case when  $S_s$  has a connected subset  $C$  containing a point on the bottom edge of  $X_s$  and a point on the left edge of  $X_s$ . We abbreviate this by saying that the parameter  $s$  has an acute crosscut. At this point we no longer care about the condition that  $s > 1/2$ . As above, we will treat the case when  $C$  is a path; the Chain Trick handles the general case.

**Lemma 7.9** *If an octagonal parameter  $s$  has an acute crosscut, then there is another octagonal parameter  $u$  which has an acute crosscut not entirely contained in  $Z_s$ .*

**Proof:** If  $s$  has an acute crosscut contained in  $Z_s$  and  $t = R(s)$  also has an acute crosscut. If the acute crosscut for  $t$  lies in  $Z_t$  we can repeat the procedure. Every one or two steps of the procedure, the distance between the endpoints of the crosscut increases by a factor of at least  $\sqrt{2}$ . So, eventually we reach a stage where the crosscut cannot lie in  $Z_s$ . ♠

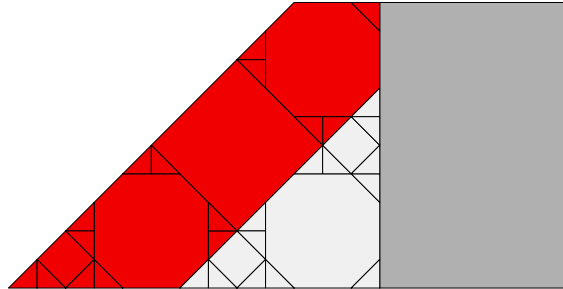
Say that an *egregious crosscut* is a connected subset of  $S_s$  which contains a point in the top edge of  $X_s$  and a point on the bottom edge of  $X_s$ .

**Lemma 7.10** *If  $s < 1/2$  has an acute crosscut which does not lie in  $Z_s$  then  $R(s)$  has an egregious crosscut.*

**Proof:** Define

$$Z_s^* = (Z_s^0 \cup R_D(Z_s^0)) \cap X_s \quad (26)$$

Figure 7.4 shows this set.



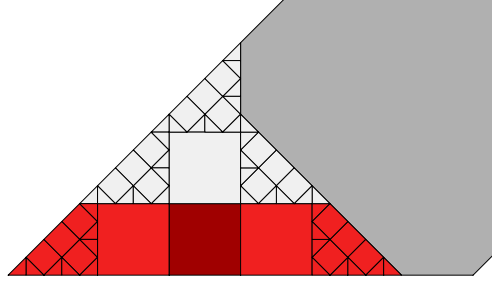
**Figure 7.4:** The set  $Z_s^*$  for  $s = 5/13$ .

$C$  starts out on the left edge of  $Z_s^*$  and must eventually reach the bottom edge of  $X_s$ . But then  $C$  must cross the right edge of  $Z_s^*$ . By symmetry, the left branch  $\phi_s^0$  of the map  $\phi_s$  from Theorem 2.5 extends to all of  $Z_s^*$ . When we pull back the maximal initial portion of  $C$  lying in  $Z_s^*$ , we get an egregious crosscut for  $t$ . ♠

**Lemma 7.11** *If  $s > 1/2$  has an acute crosscut which does not lie in  $Z_s$ , then one of  $R(s)$  or  $R^2(s)$  has an egregious crosscut.*

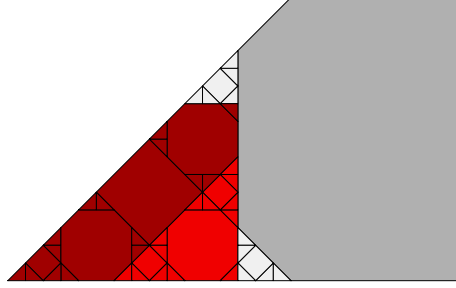
**Proof:** Let  $K$  be the layering constant for  $s$ . When  $K > 1$  we define

$$Z_s^* = (Z_s^0 \cup R_V(Z_s^0)) \cap X_s \quad (27)$$



**Figure 7.5:** The set  $Z_s^*$  for  $s = 11/13$ .

If  $C$  starts out above  $Z_s^*$  then  $C$  must cross both the top and bottom edges of  $Z_s^*$  to reach the bottom edge of  $X_s$ . If  $C$  starts out in  $Z_s^*$  but does not lie entirely in  $Z_s^0$ , then  $C$  must exit the top of  $Z_s^*$ . The problem is that  $C$  cannot penetrate through the (dark red) central tile of  $Z_s^0$ . So, in this case as well,  $C$  crosses both the top and the bottom of  $Z_s^*$ . Moreover,  $C$  must cross both sides on the same side (left or right) of the central tile of  $Z_s^0$ . Reflecting in  $V$ , we can assume that  $C$  crosses both sides of  $Z_s^*$  to the left of the central tile of  $Z_s^0$ . But then some connected subset of  $C$  lies in  $Z_s^0$  and contains points both on the top edge and the bottom edge. Now we pull back by  $\phi_s$ , as above.



**Figure 7.6:** The set  $Z_s$  for  $s = 11/13$ .

Now suppose that  $K = 1$ . Figure 7.6 shows a representative example. The red set is  $Z_s^0$ . The dark red subset is a similar copy of  $Z_t^*$ , defined in Equation 26. Call this similar copy  $\Omega$ . If  $C$  exits  $Z_s^0$  then one of two things must happen.

1.  $C$  contains points on the left and right diagonal edges of  $\Omega$ .
2.  $C$  contains points on the top edge and right diagonal edge of  $\Omega$ .

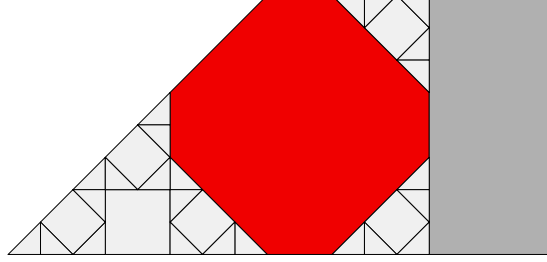
In the first case, we proceed as in Lemma 7.10, and get an egregious crosscut for  $u = R(t) = R^2(s)$ . In the second case, we also proceed as in Lemma 7.10, but we get something different: After reflecting through the origin, we get a connected set  $C' \subset S_u$  which has points on the bottom edge of  $X_u^0$  and some point lying above the line  $H$  of symmetry. (The point is that  $C$  crosses the diagonal midline of  $\Omega$ .) But then  $C'' = C' \cup R_H(C')$  is an egregious crosscut for  $u$ . ♠

Combining what we have proved so far, we get the following.

**Corollary 7.12** *If some octagonal parameter  $s$  has an acute crosscut, then some octagonal parameter  $u < 1/2$  has an egregious crosscut.*

**Lemma 7.13** *If  $s < 1/2$  has an egregious crosscut then  $t = R(s) < 1/2$  and  $t$  has an egregious crosscut.*

**Proof:** When  $s < 1/2$  and  $R(s) > 1/2$ , it follows from Theorem 2.5 (or from a direct calculation) that there is a large octagon which separates the top edge of  $X_s^0$  from the bottom edge of  $X_s^0$ . See Figure 7.7.

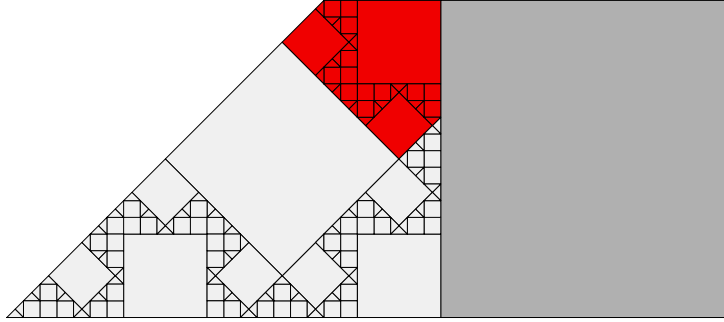


**Figure 7.7:** The large octagon.

Now we know that  $R(s) < 1/2$ . In this case,  $C$  has to cross the set

$$Z'_s = R_D(Z_s^0) \cap X_s. \quad (28)$$

$C$  starts out in the top edge of  $Z'_s$  and exits through the bottom edge.



**Figure 7.8:**  $Z'_s$  (red) for  $s = 19/52$ .

The set

$$C' = \phi_s^{-1} \circ R_D(C) \quad (29)$$

connects a point on the bottom edge of  $X_t$  with a point that lies above the line  $H$  of symmetry. In this case (as in the proof of Lemma 7.11)  $C' \cup R_H(C')$  is an egregious crosscut for  $t$ . ♠

Iterating Lemma 7.13, we see that  $R^n(s) < 1/2$  for all  $n$  when  $s$  has an egregious crosscut. So,  $s$  cannot be octagonal in this case! If we assume that Statement 3 of the Main Theorem is false, then we have reached a contradiction.

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